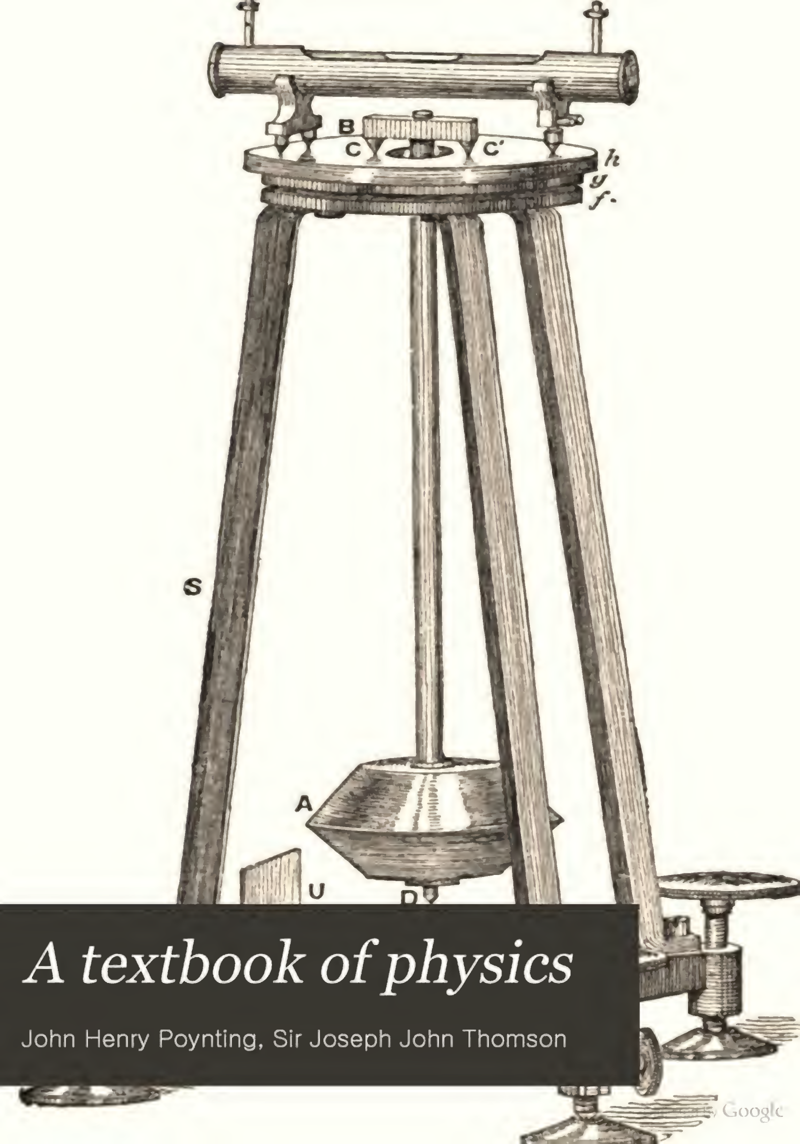

A textbook of physics

John Henry Poynting, Sir Joseph John Thomson

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A
TEXT-BOOK OF PHYSICS

PROPERTIES OF MATTER

A TEXT-BOOK OF PHYSICS

BY

J. H. POYNTING,
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A
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ROYAL INSTITUTION

PROPERTIES OF MATTER

WITH 168 ILLUSTRATIONS

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PREFACE.

THE volume now presented must be regarded as the opening one of a series forming a Text-Book on Physics, which the authors are preparing. The second volume, that on Sound, has already been issued, and the remaining volumes dealing with Heat, Magnetism and Electricity, and Light will be published in succession.

As already stated in the preface to the volume on Sound, "The Text-Book is intended chiefly for the use of students who lay most stress on the study of the experimental part of Physics, and who have not yet reached the stage at which the reading of advanced treatises on special subjects is desirable. To bring the subject within the compass thus prescribed, an account is given only of phenomena which are of special importance, or which appear to throw light on other branches of Physics, and the mathematical methods adopted are very elementary. The student who possesses a knowledge of advanced mathematical methods, and who knows how to use them, will, no doubt, be able to work out and remember most easily a theory which uses such methods. But at present a large number of earnest students of Physics are not so equipped, and the authors aim at giving an account of the subject which will be useful to students of this class. Even for the reader who is mathematically trained, there is some advantage in the study of elementary methods, compensating for their cumbrous form. They bring before us more evidently the points at which the various assumptions are made, and they render more prominent the conditions under which the theory holds good."

In the present volume the authors deal with weight, mass, gravitation, and those properties of matter which relate chiefly to change of form, such as Elasticity, Fluid Viscosity, Surface Tension, Diffusion and Solution. The molecular theory of matter has necessarily been introduced, inasmuch as investigators have almost always expressed their work in terms of that theory. But the detailed account of the theory, especially as applied to gases, will be given in the volume on Heat, in connection with the account of the phenomena which first brought it into prominence.

PREFACE TO FOURTH EDITION.

A FEW corrections have been made in this edition. The authors desire to thank the readers who have kindly pointed out errors and have enabled them to make these corrections.

J. H. P.
J. J. T.

January 1907.

CONTENTS.

CHAP. PAGE

I. WEIGHT AND MASS 1

II. THE ACCELERATION OF GRAVITY. ITS VARIATION AND THE FIGURE OF THE EARTH 7

III. GRAVITATION 28

IV. ELASTICITY 53

V. STRAIN 62

VI. STRESSES. RELATION BETWEEN STRESSES AND STRAINS 68

VII. TORSION 78

VIII. BENDING OF RODS 85

IX. SPIRAL SPRINGS 103

X. IMPACT 109

XI. COMPRESSIBILITY OF LIQUIDS 116

XII. THE RELATION BETWEEN THE PRESSURE AND VOLUME OF A GAS 124

XIII. REVERSIBLE THERMAL EFFECTS ACCOMPANYING ALTERATIONS IN STRAINS 131

XIV. CAPILLARITY 135

XV. LAPLACE'S THEORY OF CAPILLARITY 173

XVI. DIFFUSION OF LIQUIDS 182

XVII. DIFFUSION OF GASES 196

XVIII. VISCOSITY OF LIQUIDS 205

INDEX 225

PROPERTIES OF MATTER.

CHAPTER I.

WEIGHT AND MASS.

CONTENTS.—Weight—Mass—Definition of Mass—Mass proportional to Weight at the same Point—Constancy of Mass—Unit of Mass.

Introductory Remarks.—Physics is the study of the properties of matter, and of the action of one portion of matter upon another, and ultimately of the effects of these actions upon our senses. The properties studied in the various branches, Sound, Heat, Light, and Magnetism and Electricity, are for the most part easily classified under these headings. But there are other properties chiefly connected with changes in shape and relative position within a system which are grouped together as "General Properties of Matter." Among these latter properties are Elasticity, Surface Tension, Diffusion and Viscosity.

The most general properties of matter are really those studied in Statics and Dynamics: the relation between forces, when the matter acted on is in equilibrium and the motion of matter under the mutual action of the various portions of a system. But in Statics and Dynamics the recourse to experiment is so small, and when the experimental foundation is once laid the mathematical structure is so great, that it is convenient to treat these branches of Physics separately. We shall assume in this work that the reader has already studied them, and is familiar both with the conditions of equilibrium and with the simpler types of motion.

We shall, however, begin with the discussion of some questions which involve dynamical considerations. We shall show how we pass from the idea of *weight* to that of *mass*, and how we establish the doctrine of the constancy of mass. We shall then give some account of the measurement of gravity at the surface of the earth, and of the gravitation which is a property of all matter wherever situated. We shall then proceed to the discussion of those properties of matter which are perhaps best described as involving change of form.

Weight.—All matter at the surface of the earth has weight, or is pulled towards the ground. The fact that the pull is to the earth at all parts of its surface shows conclusively that it is due to the earth. Apparent exceptions, such as the rising of a balloon in air, or of a cork in water, are of course explained, not by the levity of the rising bodies, but by the greater gravity of their surroundings. Common experience with the balance shows that the ratio of the weights of two bodies is constant wherever they are weighed, so long as they are both weighed at the same point. Common experience shows too that the ratio is the same, however the bodies be turned about on the scale-pan of the balance.

The balance does not tell us anything as to the constancy of weight of a given body, but only as to the constancy of ratio; for if the weights of different bodies varied, and the variation was always in the same ratio, the balance would fail to indicate it. But here experiments with pendulums supplement our knowledge. A given pendulum at a constant temperature and in a fixed position has, as nearly as we can observe, the same time of swing from day to day and from year to year. This implies that the pull of the earth on the bob is constant—*i.e.*, that the weight at the same place remains the same.

This constancy of weight of a body at the same point appears to hold whatever chemical or physical changes the matter in it may undergo. Experiments have been made on the weight of sealed tubes containing two substances which were at first separated, and which were then mixed and allowed to form new chemical compounds. The tubes were weighed before and after the mixture of their contents. But though Landolt[*] and Heydweiller[†] have thought that the variations which they observed were real and not due to errors of experiment, Sanford and Ray[‡] have made similar experiments, and considered that the variations were observational errors. Where variations have been observed they are so minute and so irregular that we cannot as yet assume that there is any change in weight.

Again, temperature does not appear to affect weight to any appreciable extent. It is extremely difficult to make satisfactory weighings of a body at two different temperatures. Perhaps the best evidence of constancy is obtained from the agreement in the results of different methods of measuring liquid expansion. In Dulong and Petit's U-tube method of determining the expansion of mercury, two unit columns have different heights but equal weights, and it is assumed that the cold column would expand into the hot column without change of weight. But in the dilatometer method nearly the whole expansion is directly measured, and only the small expansion of the envelope, measured by assuming the expansion of mercury, introduces the assumption of constancy of weight with change of temperature. The close agreement of the two methods shows that there is no large variation of weight with temperature.

We may probably conclude that, up to the limit of our present powers of measurement, the weight of a body at a given point is constant under all conditions.

But when we test the weight at different points this constancy no longer holds. The common balance used in the ordinary way fails to show variation, since both pans are equally affected.

But very early in the history of the pendulum, as we shall show in the next chapter, experiments proved that the seconds pendulum had different lengths at different places, or that the same pendulum had different times of swing at different places. In other words, the weight of the bob varied. Thus a body is about 1 in 300 heavier at London than at the Equator.

---

[*] *Zeit. f. Physik. Chem.*, xii. 1, 1894.

[†] *Zeit. f. Physik.*, August 25, 1900, p. 527.

[‡] *Phys. Rev.*, v. 1897, p. 247.

As early as 1662 an experiment was made by Dr. Power* in which a variation of weight with change of level over the same point was looked for. A body was weighed by a fixed balance, being first placed in the scale-pan and then hung far below the same pan by a string. The experiment was repeated by Hooke, and later by others, but the variation was quite beyond the range of observation possible with these early experimenters, and the results they obtained were due to disturbances in the surroundings. The first to show that the balance could detect a variation was von Jolly (chap. iii. p. 41), who in 1878 described an experiment in which he weighed a kilogramme on a balance 5·5 metres above the floor and then hung the kilogramme by a wire so that it was near the floor. He detected a gain in the lower position of 1·5 mgm. Later he repeated the experiment on a tower, a 5 kgm. weight gaining more than 31 mgms. between the top of the tower and a point 21 metres below. More recently Richarz and Krigar-Menzel found a variation in the weight of a kilogramme when lowered only 2 metres (chap. iii. p. 42.)

The evidence then is convincing that the weight of a body varies from point to point on the earth's surface, and also varies with its distance above the same point.

The question now arises—Is there any measurable quality of matter which remains the same wherever it is measured? Experiment shows that there is constancy in that which is termed the *mass of matter*.

Mass.—Without entering into any discussion of the most appropriate or most fundamental method of measuring force, we shall assume that we can measure forces exerted by bent and stretched springs and similar contrivances independently of the motion they produce. We shall assume that, when a given strain is observed in a spring, it is acting with a definite force on the body to which it is attached, the force being determined by previous experiments on the spring. Let us imagine an ideal experiment in which a spring is attached to a certain body, which it pulls horizontally, under constraint free from friction. Let the spring be always stretched to a given amount as it pulls the body along, so acting on it with constant force. Then all experiments and observations go to show that the body will move with the same constant acceleration wherever the experiment is made. This constancy of acceleration under a given force is expressed by saying that the mass of the body is constant. Though the experiment we have imagined is unrealisable, actual experiments on the same lines are made for us by good chronometers. The balance-wheel of a chronometer moves to and fro against the resistance of the hair-spring, and its acceleration is very accurately the same for the same strain of the spring at the same temperature in different latitudes. The weight of the balance-wheel decreases by 3 in 1000 if the chronometer is carried from London to the Equator. If the acceleration under given force increased in the same ratio the rate of the chronometer would change by 3 in 2000, or by two minutes per day, and the chronometer would be useless for determinations of longitude. Again, a tuning-fork, making, say, 256 vibrations per second at Paris at 16° will have very accurately the same frequency at the same temperature wherever tested. The same portion of matter in the prongs has the same acceleration for the same strain and, presumably, for the same force all the world over.

---

* Mackenzie, *The Laws of Gravitation*, p. 2.

This constancy of acceleration of a given body under given force holds true likewise whatever the nature of the body exerting the force may be—*i.e.*, whether it be a bent spring, a spiral spring, air pressing, a string pulling, and so on.

Further experiment shows that the acceleration of a given body is proportional to the force acting on it. Thus, in a very small vibration of a pendulum the fraction of the weight of the bob tending to restore it to its central position is proportional to the displacement, and the simple harmonic type of the motion with its isochronism shows at once that the acceleration is proportional to the displacement, and therefore to the force acting. When a body vibrates up and down at the end of a spiral spring we again have simple harmonic motion with acceleration proportional to the distance from the position of equilibrium. The variation in the force exerted by the spring is also proportional to this distance, or acceleration is proportional to force acting. Indeed, elastic vibrations with their isochronism go, in general, to prove this proportionality. If, then, we accept the view that we can think of forces acting on bodies as being measurable independently of the motion which they produce—measurable, say, by the strain of the bodies acting—we have good experimental proof that a given portion of matter always has equal acceleration under equal force, and that the accelerations under different forces are proportional to the forces acting upon it.

We can now go a step farther and use the accelerations to compare different masses.

Definition of Mass.—*The masses of bodies are proportional to the forces producing equal accelerations in them.*

An equivalent statement is, that the masses are inversely as the acceleration produced by equal forces. It follows from our definition that, if equal accelerations are observed in different bodies, then the masses are proportional to the forces acting.

Observation and experiment further enable us to say that:

The masses of bodies are proportional to their weights at the same point. To prove this it is only necessary to show that all bodies have equal acceleration at the same place when acted on by their weights alone—to show, in fact, that the quantity always denoted by g is constant at the same place.

A very simple though rough experiment to prove this consists in tying a piece of iron and a piece of wood to the two ends of a thread and putting the thread across a horizontal ring so that the two weights depend at the same height above the floor. The thread is now burnt in the middle of the ring and the iron and wood begin to fall at the same instant. They reach the floor so nearly together that only a single bump is heard. If the surfaces presented to the air are very different the air resistance may interfere with the success of the experiment. But the more the air resistance is eliminated the more nearly is the time of fall the same. Thus, if a penny and a sheet of paper are placed on a board some height above the floor, and if the board is suddenly withdrawn, the penny falls straight while the paper slowly flutters down. Now crumple up the paper into a little ball and repeat the experiment, when the two reach the ground as nearly as we can observe together.

Newton (*Principia*, Book III., Prop. 6) devised a much more accurate form of the experiment, using the pendulum, in which any difference of acceleration would be cumulative, and suspending in succession equal weights of various kinds of matter. He says (Motte's translation):

> "It has been, now of a long time, observed by others, that all sorts of heavy bodies (allowance being made for the inequality of retardation, which they suffer from a small power of resistance in the air) descend to the Earth *from equal heights* in equal times; and that equality of times we may distinguish to a great accuracy, by the help of pendulums. I tried the thing in gold, silver, lead, glass, sand, common salt, wood, water, and wheat. I provided two wooden boxes, round and equal. I filled the one with wood, and suspended an equal weight of gold (as exactly as I could) in the centre of oscillation of the other. The boxes hanging by equal threads of eleven feet, made a couple of pendulums perfectly equal in weight and figure, and equally receiving the resistance of the air. And placing the one by the other, I observed them to play together forwards and backwards, for a long time, with equal vibrations. And therefore the quantity of matter in the gold (by Cor. 1 and 6, prop. 24, book 2) was to the quantity of matter in the wood, as the action of the motive force (or *vis motrix*) upon all the gold, to the action of the same upon all the wood; that is, as the weight of the one to the weight of the other. And the like happened in the other bodies. By these experiments, in bodies of the same weight, I could manifestly have discovered a difference of matter less than a thousandth part of the whole, had any such been."

Newton here uses "quantity of matter" where we should now say "mass." Bessel (*Berlin Abh.*, 1830, *Ann. Pogg.*, xxv. 1832, or *Mémoires relatifs à la Physique*, v. p. 71) made a series of most careful experiments by Newton's method, fully confirming the conclusion that weight at the same place is proportional to mass.

Constancy of Mass.—The experiments which have led to the conclusion that weight at the same place is constant now gain another significance. They show that the mass of a given portion of matter is constant, whatever changes of position, of form, or of chemical or physical condition it may undergo.

When we "weigh" a body by the common balance, say, by the counterpoise method, we put it on the pan, counterpoise it, and then replace it by bodies from the set of "weights" having an equal weight.

But our aim is not to find the weight of the body, the pull of the earth on it. We use the equality of weight possessed by equal masses at the same point of the earth's surface to find its mass. In buying matter by weight we are not ultimately concerned with weight but with mass, and we expect the same mass in a pound of it whether we buy in London or at the Equator. A set of weights is really a set of masses, and when we use one of them we are using it as a mass through its weight.

Unit of Mass.—We can make a definite unit of mass by fixing on some piece of matter as the standard and saying that it contains one unit or so many units. So long as we are careful that no portion of the standard piece of matter is removed and that no addition is made to it, such a unit is both definite and consistent.

In this country the unit of mass for commercial purposes is the piece of platinum kept at the Standards Office at Westminster, marked "P.S. 1844 1 lb." and called the Imperial Avoirdupois Pound. But for scientific purposes all over the world the unit of mass is the gramme, the one-thousandth part of the mass of the piece of platinum-iridium called the "Kilogramme-International," which is kept at Paris. Copies of this kilogramme, compared either with it or with previous copies of it, are now distributed through the world, their values being known to less, perhaps, than 0·01 mgm. For example, the copy in the Standards Office at Westminster is certified to be

$$1{\cdot}000000070 \text{ kgm.}$$

with a probable error of 2 in the last place.

According to a comparison carried out in 1883, the Imperial pound contains

$$453{\cdot}5924277 \text{ grammes,}$$

though Parliament enacted in 1878 that the pound contained

$$453{\cdot}59245 \text{ grammes.}$$

Of course one piece of matter only can be the standard in one system of measurements, and the enactment of 1878 only implies that we should use a different value for the kilogramme in England from that used in France. The difference is, however, quite negligible for commercial purposes.

CHAPTER II.

THE ACCELERATION OF GRAVITY. ITS VARIATION AND THE FIGURE OF THE EARTH.

Contents.—Early History—Pendulum Clock—Picard's Experiments—Huygens' Theory—Newton's Theory and Experiments—Bouguer's Experiments—Bernouilli's Correction for Arc—Experiments of Borda and Cassini—Kater's Convertible Pendulum—Bessel's Experiments and his Theory of the Reversible Pendulum—Repsold's Pendulum—Yielding of the Support—Defforges' Pendulum—Variation of Gravity over the Earth's Surface—Richer—Newton's Theory of the Figure of the Earth—Measurements in Sweden and Peru—Bouguer's Correction to Sea-level—Clairaut's Theorem—Kater and Sabine—Invariable Pendulum—Airy's Hydrostatic Theory—Faye's Rule—Indian Survey—Formula for g in any Latitude—Von Sterneck's Half-second Pendulums—His Barymeter—Gravity Balance of Threlfall and Pollock.

We shall describe in this and the following chapter the methods of measuring two quantities; the acceleration of falling bodies due to the earth, at its surface (the quantity always denoted by g); and the acceleration due to unit mass at unit distance (the quantity known as the *gravitation constant* and denoted by G). The two may be measured quite independently, but yet they are closely related in that g is the measure of a particular case of gravitation, while G is the expression of its general measure. The two together enable us to find the mass and therefore the mean density of the earth.

The Acceleration of Gravity.*—We shall briefly trace the history of the methods which have been used in measuring g, for in so doing we can set forth most clearly the difficulties to be overcome and realise the exactitude with which the measurement can now be made. We shall then give some account of the experiments made to determine the variations of gravity and the use of the knowledge so gained to determine the shape of the earth.

Early History.—The first step in our knowledge of the laws of falling bodies was taken about the end of the sixteenth century, when Stevinus, Galileo, and their contemporaries were laying the foundations of the modern knowledge of mechanics. Stevinus, the discoverer of the Triangle of Forces and of the theory of the Inclined Plane, and Galileo, the founder of Dynamics, were both aware that the doctrine then held that bodies fall with rapidity proportional to their weight was quite false, and they asserted that under the action of their weight alone all bodies would fall at equal rates. They pointed out that the different rates actually observed were to be ascribed to the resistance of the air, which has a greater effect on the movement of light than of heavy bodies of equal size. Galileo made a celebrated experiment to verify this fact by dropping bodies of different weights from the top of the Leaning Tower of Pisa, and showing that they reached the ground in the same time. The air-pump was not yet invented, so that the later verification by the "guinea and feather" was not then possible. But Galileo did not stop with this experiment. He made the progress of dynamics possible by introducing the conception of equal additions of velocity in equal times—the conception of uniform acceleration. His first idea was that a constant force would give equal additions of velocity in equal distances traversed, but investigation led him to see that this idea was untenable, and he then enunciated the hypothesis of equal additions in equal times. He showed that, by this hypothesis, the distance traversed is proportional to the square of the time. Not content with mere mathematical deductions, he made experiments on bodies moving down inclined planes, and demonstrated that the distances traversed were actually proportional to the squares of the times—*i.e.*, that the acceleration was uniform. By experiments with pendulums falling through the arc of a circle to the lowest point, and then rising through another arc, he concluded that the velocity acquired in falling down a slope depends only on the vertical height fallen through and not upon the length of the slope, or, as we should now put it, that the acceleration is proportional to the cosine of the angle of the slope with the vertical. He thus arrived at quite sound ideas on the acceleration of falling bodies and on its uniformity, and from his inclined plane experiments could have obtained a rough approximation to the quantity we now denote by g. But Galileo had no accurate method of measuring small periods of time in seconds. The pendulum clock was not as yet invented, and he made merely relative measurements of the time intervals by determining in his experiments the quantity of water which flowed through a small orifice of a vessel during each interval.

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* A collection of the most important original papers on the pendulum constitutes vols. iv. and v. of *Mémoires relatifs à la Physique*. It is prefaced by an excellent history of the subject by M. Wolf, and contains a bibliography. The fifth volume of *The G. T. Survey of India* consists of an account of the pendulum operations of the survey, with some important memoirs. In the *Journal de Physique*, vii. 1888, are three important articles by Commandant Defforges on the theory of the pendulum, concluding with an account of his own pendulum. The description given in this chapter is based on these works.

To Galileo we also owe the foundation of the study of pendulum vibrations. The isochronism of the pendulum had been previously observed by others, but Galileo rediscovered it for himself, and showed by further experiment that the times of vibration of different simple pendulums are proportional to the square roots of their lengths. He also used the pendulum to determine the rate of beating of the pulse and recognised the possibility of employing it as a clock regulator. He did not publish his ideas on the construction of a pendulum clock, and they were only discovered among his papers long after his death.

From Galileo, therefore, we derive the conception of the appropriate quantity to measure in the fall of bodies, the acceleration, and to him we owe the instrument which as a free pendulum gives us the acceleration of fall, and, as a clock regulator, provides us with the best means of determining the time of fall.

Soon after Galileo's death, Mersenne made, in 1644, the first determination of the length of a simple pendulum beating seconds, and a little later he suggested as a problem the determination of the length of a simple pendulum equivalent to a given compound pendulum.

Pendulum Clock.—But it was only with the invention of the pendulum clock by Huygens in 1657 that the second became an interval of time measurable with consistency and ease. At once the new clock was widely used. Its rate could easily be determined by star observations, and determinations of the length of the seconds pendulum by its aid became common.

Picard's Experiment.—In 1669 Picard determined this length at Paris, using a copper ball an inch in diameter suspended by an aloe fibre from jaws. This suspension was usual in early work, the aloe fibre being unaffected to any appreciable extent by moisture. Picard's value was 36 inches $8\frac{1}{2}$ lines Paris measure. The Paris foot may be taken as $\frac{114}{107}$ or 1·065 English feet, and there are 12 lines to the inch, so that the length found was 39·09 English inches. Picard states that the value had been found to be the same at London and at Lyons.

Huygens' Theory.—In 1673 Huygens propounded the theory of the cycloidal pendulum, proving its exact isochronism, and he showed how to construct such a pendulum by allowing the string to vibrate between cycloidal cheeks. He determined the length beating seconds at Paris, confirming Picard's value, and from the formula which we now put in the form $g=\pi^2 l$ he found $\frac{g}{2}$ the distance of free fall in one second, the quantity which was at first used, instead of the full acceleration we now employ. His value was 15 ft. 1 in. $1\frac{7}{9}$ lines, Paris measure, which would give $g=32{\cdot}16$ English feet.

Huygens at the same time gave the theory of uniform motion in a circle and the theory of the conical pendulum, and above all in importance he founded the study of the motion of bodies of finite size by solving Mersenne's problem and working out the theory of the compound pendulum. He discovered the method of determining the centre of oscillation and showed its interchangeability with the centre of suspension.

Newton's Theory and Experiments.—Newton in the *Principia* made great use of the theory of the pendulum. He there for the first time made the idea of mass definite, and by his pendulum experiments (*Principia*, sect. vi., Book II., Prop. 24), he proved that mass is proportional to weight. He used pendulums too, to investigate the resistance of the air to bodies moving through it, and repeated the pendulum experiments of Wren and others, by which the laws of impact had been discovered. But his great contribution to our present subject was the demonstration, by means of the moon's motion, that gravity is a particular case of gravitation and acts according to the law of inverse squares, the attracting body being the earth. In Book III., Prop. 4, he calculates the acceleration of the moon towards the earth and shows that, starting from rest with this acceleration, it would fall towards the earth 15 ft. 1 in. $1\frac{4}{9}$ lines (Paris) in the first minute. If at the surface of the earth 60 times nearer the acceleration is 60^2 times greater the same distance would here be fallen through in one second, a distance almost exactly that obtained by Huygens' experiments.

In a later proposition (37) he returns to this calculation, and now, assuming the law of inverse squares to be correct, he makes a more exact determination of the moon's acceleration, and from it deduces the value of gravity at the mean radius of the earth in latitude 45°. Then by his theory of the variation of gravity with latitude, of which we shall give some account below, he finds the value at Paris. He corrects the value thus found for the centrifugal force at Paris and (in Prop. 19) for the air displaced, which he takes as $\frac{1}{11000}$ of the weight of the bob used in the pendulum experiments, and finally arrives at 15 ft. 1 in. $1\frac{1}{2}$ lines (Paris), differing from Huygens' value by about 1 in 7500.

Bouguer's Experiments.—Though Newton was thus aware of the need of the correction for the buoyancy of the air, it does not appear to have been applied again until Bouguer made his celebrated experiments in the Andes in 1737. These are especially interesting in regard to the variations of gravity, but we may here mention some important points to which Bouguer attended. While his predecessors probably altered the length of the pendulum till it swung seconds as exactly as could be observed, Bouguer introduced the idea of an "invariable pendulum," making it always of the same length and observing how long it took to lose so many vibrations on the seconds clock. For this purpose the thread of the pendulum swung in front of a scale, and he noted the time when the thread moved past the centre of the scale at the same instant that the beat of the clock was heard. Here we have an elementary form of the "method of coincidences," to be described later. He used, not the measured length from the jaw suspension to the centre of the bob, which was a double truncated cone, but the length to the centre of oscillation of the thread and bob, and he allowed for change of length of his measuring-rod with temperature. He also assured himself of the coincidence of the centre of figure with the centre of gravity of the bob by showing that the time of swing was the same when the bob was inverted. He determined the density of the air by finding the vertical height through which he must carry a barometer in order that it should fall one line, and he thus estimated the density of the air on the summit of Pichincha at $\frac{1}{11000}$ that of the copper bob of his pendulum. Applying these corrections to his observations he calculated the length of the seconds pendulum *in vacuo*.

Correction for Arc.—In 1747, D. Bernouilli showed how to correct the observed time of vibration to that for an infinitely small arc of swing. The observed time is to a first approximation longer than that for an infinitely small arc in the ratio $1+\frac{a^2}{16}$ where a is the amplitude of the angle of swing. The correction has to be modified for the decrease in amplitude occurring during an observation.

Experiments of Borda and Cassini.—The next especially noteworthy experiments are those by Borda and Cassini made at Paris in 1792 in connection with the investigations to determine a new standard of length, when it was still doubtful whether the seconds pendulum might not be preferable to a unit related to the dimensions of the earth. The form of pendulum which they used is now named after Borda. It consisted of a platinum ball nearly $1\frac{1}{2}$ inches in diameter, hung by a fine iron wire about 12 Paris feet long. It had a half-period of about two seconds The wire was attached at its upper end to a knife edge—the advantages of a knife-edge suspension having been already recognised—and the knife edge and wire-holder were so formed that their time of swing alone was the same as that of the pendulum. In calculating the moment of inertia, they could therefore be left out of account. At the lower end the wire was attached to a shallow cup with the concavity downwards, and the ball exactly fitted into this cup, being made to adhere to it by a little grease. The ball could therefore be easily and exactly reversed without altering the pendulum length, and any non-coincidence of centre of gravity and centre of figure could be eliminated by taking the time of swing for each position of the ball. The pendulum was hung in front of a seconds clock, with its bob a little below the clock bob, and on the latter was fixed a black paper with a white X-shaped cross on it. The vibrations were watched through a telescope from a short distance away, and a little in front of the pendulum was a black screen covering half the field. When the pendulums were at rest in the field the edge of this screen covered half the cross and half the wire. When the swings were in progress the times were noted at which the pendulum wire just bisected the cross at the instant of disappearance behind the screen. This was a "coincidence," and, since the clock bob made two swings to one of the pendulum, the interval between two successive "coincidences" was the time in which the clock gained or lost one complete vibration or two seconds on the wire pendulum. The exact second of a coincidence could not be determined but only estimated, as for many seconds the wire and cross appeared to pass the edge together. But the advantage of the method of coincidences was still preserved, for it lies in the fact that if the uncertainty is a small fraction of the interval between two successive coincidences the error introduced is a very much smaller fraction of the time of vibration. For, suppose that the wire pendulum makes n half swings while the clock makes $2n+2$. If the clock beats exact seconds the time of vibration of the wire pendulum is

$$t = \frac{2n+2}{n} = 2\left(1 + \frac{1}{n}\right)$$

If there is a possible error in the determination of each of two successive coincidences of m seconds, or at the most of $2m$ in the interval of $2n+2$ seconds, the observed time might be

$$t = 2\left(1 + \frac{1}{n \pm m}\right) = 2\left\{1 + \frac{1}{n}\left(1 \mp \frac{m}{n}\right)\right\} = 2\left(1 + \frac{1}{n} \mp \frac{m}{n^2}\right)$$

In one case Borda and Cassini employed an interval of $2n = 3000$ seconds, and found an uncertainty not more than 30 seconds for the instant of coincidence. Thus

$$\frac{m}{n^2} = \frac{30}{1500^2} = \frac{1}{75000}$$

Now, as they observed for about four hours, or for five intervals in succession, the error was reduced to $\frac{1}{5}$, or $\frac{1}{375000}$ of the value of t. Practically the method of coincidences determined the time of vibration of the pendulum in terms of the clock time with sufficient accuracy, and the responsibility for error lay in the clock. The pendulum was treated as forming a rigid system, and the length of the equivalent ideal simple pendulum was calculated therefrom. Corrections were made for air displaced, for arc of swing, and for variations in length with temperature.

The final value obtained was: Seconds pendulum at Paris = 440·5593 lines (Paris). As the metre = 443·296 Paris lines, this gives 993·53 mm., and, corrected to sea-level, it gives 993·85 mm.

Kater's Convertible Pendulum.—The difficulties in measuring the length and in calculating the moment of inertia of the wire-suspended or so-called simple pendulum led Prony in 1800 to propose a pendulum employing the principle of interchangeability of the centres of oscillation and suspension. The pendulum was to have two knife edges turned inwards on opposite sides of the centre of gravity, so that it could be swung from either, and was to be so adjusted that the time of swing was the same in both cases. The distance between the knife edges would then be the length of the equivalent simple pendulum. Prony's proposal was unheeded by his contemporaries, and the paper describing it was only published eighty years later.*

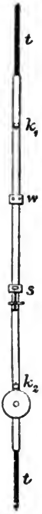

FIG. 1.—Kater's Convertible Pendulum.

In 1811, Bohnenberger made the same proposal, and again in 1817 Captain Kater independently hit on the idea, and for the first time carried it into practice, making his celebrated determination of g at London with the form of instrument since known as "Kater's convertible pendulum." This pendulum is shown in Fig. 1. On the rod are two adjustable weights, w and s. The larger weight w is moved about until the times of swing from the two knife edges k_1 k_2 are nearly equal, when it is screwed in position. Then s is moved by means of a screw to make the final adjustment to equality. Kater determined the time of vibration by the method of coincidences, his use of it being but slightly different from that of Borda. A white circle on black paper was fastened on the bob of the clock pendulum; the convertible pendulum was suspended in front of the clock, and when the two were at rest the tail-piece t of the former just covered the white circle on the latter as viewed by a telescope a few feet away. A slit was made in the focal plane of the eyepiece of the telescope just the width of the images of the white patch and of the pendulum tail. A coincidence was the instant during an observation at which the white circle was quite invisible as the two pendulums swung past the lowest point together. A series of swings were made, first from one knife edge and then from the other, each series lasting over four or five coincidences, the coincidence interval being about 500 seconds. The fine weight was moved after each series till the number of vibrations per twenty-four hours only differed by a small fraction of one vibration whichever knife edge was used, and then the difference was less than errors of observation, for the time was sometimes greater from the one, sometimes greater from the other. The mean time observed when this stage was reached was corrected for amplitude, and then taken as the time of the simple pendulum of length equal to the distance between the knife edges, this distance being carefully measured. A correction was made for the air displaced on the assumption that gravity was diminished thereby in the ratio of weight of pendulum in air to weight of pendulum in vacuo. The value was then corrected to sea-level. The final value of the length of the seconds pendulum at sea-level in the latitude of London was determined to be 39·13929 inches.†

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* *Mémoires relatifs à la Physique*, iv. p. 65.

† The experiments are described in a paper in the *Phil. Trans.* for 1818 "An

Bessel's Experiments and his Theory of the Reversible Pendulum.—In 1826 Bessel made experiments to determine the length of the seconds pendulum at Koenigsberg. He used a wire-suspended pendulum, swung first from one point and then from another point, exactly a "Toise of Peru"* higher up, the bob being at the same level in each case. Assuming that the pendulums are truly simple, it will easily be seen that the difference in the squares of the times is the square of the time for a simple pendulum of length equal to the difference in lengths, and therefore the actual length need not be known. But the practical pendulum departs from the ideal simple type, and so the actual lengths have to be known. As, however, they enter into the expression for the difference of the squares of the times, with a very small quantity as coefficient, they need not be known with such accuracy as their differences. Bessel took especial care that this difference should be accurately equal to the toise. At the upper end, in place of jaws or a knife edge, he used a horizontal cylinder on which the wire wrapped and unwrapped. He introduced corrections for the stiffness of the wire and for the want of rigidity of connection between bob and wire. The necessity for the latter correction was pointed out by Laplace, who showed that the two, bob and wire, could not move as one piece, for the bob acquires and loses angular momentum around its centre of gravity, which cannot be accounted for by forces passing through the centre, such as would alone act if the line of the wire, produced, always passed through the centre. In reality the bob turns through a slightly greater angle than the wire, so that the pull of the wire is now on one side and now on the other side of the centre of gravity. The correction is, however, small if the bob has a radius small in comparison with the length of the wire.

If l is the length of the wire, r the distance of the centre of gravity of the bob from the point at which the wire is attached to it, and κ the radius of gyration of the bob about an axis through the centre of gravity; then, neglecting higher powers than κ^4, the equivalent simple pendulum can be shown to be

$$l + r + \frac{\kappa^2}{l + r} + \frac{l\kappa^4}{r(l + r)^3},$$

the last term being due to the correction under consideration. As an illustration, suppose the bob is a sphere of 1 inch radius and the wire is 38 inches long; then the equivalent simple pendulum in inches is $39 + \cdot 010256 + \cdot 000102$, and the last term, 1/400000 of the whole length, need only to be taken into account in the most accurate work.

Bessel also made a very important change in the air correction. The effect of the air on the motion may be separated into three parts—

(1) The buoyancy, the weight of the pendulum being virtually decreased by the weight of the air which it displaces.

(2) The flow of the air, some of the air moving with the pendulum, and so virtually increasing its mass.

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account of experiments for determining the length of the pendulum vibrating seconds in the latitude of London," and in a paper in the *Phil. Trans.* for 1819, "Experiments for determining the variations in the length of the pendulum vibrating seconds," Kater applies further corrections and gives the above value.

* The "Toise of Peru" was a standard bar at the Paris Observatory, 6 Paris feet or about 1949 millimetres long.

(3) The air drag, a viscous resistance which comes into play between the different layers of air, moving at different rates, a resistance transmitted to the pendulum.

As far back as 1786 Du Buat had pointed out the existence of the second effect, and had made experiments with pendulums of the same length and form, but of different densities, to determine the extra mass for various shapes. Bessel, not knowing Du Buat's work, reinvestigated the matter, and again by the same method determined the virtual addition to the mass for various shapes, and among others for the pendulum he used.

The viscous resistance was first placed in its true relation by Stokes' investigations on Fluid Motion in 1847. In pendulum motion we may regard it as tending to decrease the amplitude alone, for the effect on the time of vibration is inappreciable. We may represent its effect by introducing a term proportional to the velocity in the equation of motion, which thus becomes

$$\ddot{\theta} + \nu\dot{\theta} + \mu\theta = 0$$

The solution of this is $\theta = Ae^{-\frac{\nu t}{2}} \cos\left\{\sqrt{\mu - \frac{\nu^2}{4}}\,t - a\right\}$

where A and a are constants.

The period is $T = \dfrac{2\pi}{\sqrt{\mu - \frac{\nu^2}{4}}}$ where ν depends on the viscosity.

Approximately $T = \dfrac{2\pi}{\sqrt{\mu}}\left(1 + \dfrac{\nu^2}{8\mu}\right)$, or the time is increased by the viscosity in the ratio $1 + \dfrac{\nu^2}{8\mu} : 1$,

or since $\mu = \dfrac{4\pi^2}{T^2}$ (nearly), in the ratio $1 + \dfrac{\nu^2 T^2}{32\pi^2} : 1$.

To see the order of this alteration, suppose that $\rho_1\ \rho_2$ represent two succeeding amplitudes on opposite sides of the centre—*i.e.*, values for which

$\dot{\theta} = 0$, or $\cos\left(\sqrt{\mu - \frac{\nu^2}{4}}\,t - a - \cos^{-1}\frac{\nu}{2\mu}\right) = 0$; then $\frac{\rho_1}{\rho} = e^{\frac{\nu T}{4}}$ or, taking logarithms,

$$\log \frac{\rho_1}{\rho_2} = \lambda = \frac{\nu T}{4}$$

Now in one of Kater's experiments the arc of swing decreased in about 500 seconds from 1·41° to 1·18°, or in the ratio 1·195 : 1.

Then $\left(\dfrac{\rho_1}{\rho_2}\right)^{500} = 1{\cdot}195$ and $500\lambda = \log_e 1{\cdot}195 = 0{\cdot}178$

whence $\lambda = {\cdot}000356$ and $\dfrac{\nu^2 T^2}{32\pi^2} = \dfrac{\lambda^2}{2\pi^2} = 6 \times 10^{-9}$ about.

In Borda's pendulum the effect was about the same—*i.e.*, one that is practically quite negligible.

Bessel also used the pendulum to investigate afresh the correctness of Newton's proof that mass is proportional to weight, carrying out a series of experiments which still remain the best on the subject. But Bessel's chief contribution to gravitational research consisted of his theory of the "reversible pendulum." He showed that if a pendulum were made symmetrical in external form about its middle point, but loaded at one end, to lower the centre of gravity, and provided with two knife edges, like Kater's pendulum, one very nearly at the centre of oscillation of the other, the length of the seconds pendulum could be deduced from the two times without regard to the air effects. Laplace had shown that the knife edges must be regarded as cylinders, and not mere lines of support. Bessel showed, however, that if the knife edges were exactly equal cylinders their effect was eliminated by the inversion, and that if they were different cylinders their effect was eliminated by interchanging the knife edges, and again determining the times from each—the "erect" and "inverted" times as we may conveniently term them.

We shall consider these various points separately.

In the first place, Bessel showed that it was unnecessary to make the erect and inverted times exactly equal. For if T_1 and T_2 be these times, if h_1 h_2 be the distances of the centre of gravity from the two knife edges, and if κ be the radius of gyration round an axis through the centre of gravity, the formula for the compound pendulum gives

$$\frac{g}{4\pi^2}T_1^2 = \frac{h_1^2+\kappa^2}{h_1} \qquad \frac{g}{4\pi^2}T_2^2 = \frac{h_2^2+\kappa^2}{h_2}$$

Multiply respectively by h_1, h_2, subtract and divide by $h_1 - h_2$ and we have

$$\frac{g}{4\pi^2}\,\frac{h_1T_1^2 - h_2T_2^2}{h_1 - h_2} = h_1 + h_2$$

Let us put
$$\frac{h_1T_1^2 - h_2T_2^2}{h_1 - h_2} = T^2$$

We shall term T the *computed time*. We see that it is the time corresponding to a length of simple pendulum $h_1 + h_2$. It may be expressed in a more convenient form, thus:

Let $\tau^2 = \dfrac{T_1^2 + T_2^2}{2}$ and $a^2 = \dfrac{T_1^2 - T_2^2}{2}$,

then $T_1^2 = \tau^2 + a^2$, $T_2^2 = \tau^2 - a^2$, and substituting in T^2 we get

$$T^2 = \frac{h_1T_1^2 - h_2T_2^2}{h_1 - h_2} = \tau^2 + a^2\frac{h_1+h_2}{h_1-h_2} = \frac{T_1^2+T_2^2}{2} + \frac{T_1^2-T_2^2}{2}\,\frac{h_1+h_2}{h_1-h_2}$$

Now $h_1 + h_2$ is measurable with great exactitude, but h_1 and h_2, and therefore $h_1 - h_2$, cannot be determined with nearly such accuracy. The method of measuring them consists in balancing the pendulum in horizontal position on a knife edge and measuring the distance of the balancing knife edge from each end knife edge. But the formula shows that it is not necessary to know $h_1 - h_2$ exactly, for it only occurs in the coefficient of $T_1^2 - T_2^2$, which is a very small fraction of $T_1^2 + T_2^2$. Knowing, then, $h_1 + h_2$ exactly and $h_1 - h_2$ approximately, we can compute the time corresponding to

$h_1 + h_2$ from the times in the erect and inverted positions and avoid the troublesome series of trials which Kater made before obtaining exact equality for them from each knife edge.

Now let us consider the air effect. Take first the erect position of the pendulum. We may represent the buoyancy by an upward force applied at the centre of gravity of the displaced air, and equal to its weight mg. Let this centre of gravity be distant s from the centre of suspension.

The mass of air flowing with the pendulum will have no effective weight, since it is buoyed up by the surrounding air. It is merely an addition to the mass moved and serves to increase the moment of inertia of the pendulum. Let us represent it by the addition of a term $m'd^2$ when the pendulum is erect.

Then we have $\frac{gT_1^2}{4\pi^2} = \frac{M(h_1^2+\kappa^2)+m'd^2}{Mh_1 - ms} = \frac{M(h_1^2+\kappa^2)+m'd^2}{Mh_1}\left(1+\frac{ms}{Mh_1}\right)$

$$= \frac{h_1^2+\kappa^2}{h_1} + \frac{h_1^2+\kappa^2}{h_1}\,\frac{ms}{Mh_1} + \frac{m'd^2}{Mh_1}$$

neglecting squares and products of $\frac{m}{M}$ and $\frac{m'}{M}$, since in practice these quantities are of the order 10^{-4}.

Now invert and swing from an axis near the centre of oscillation. The value of m is the same, but its centre of gravity may be at a different distance from the new suspension, say s'. The air moving may be different, so that we must now put $m''d^2$ instead of $m'd^2$. We have then

$$\frac{gT_2^2}{4\pi^2} = \frac{h_2^2+\kappa^2}{h_2} + \frac{h_2^2+\kappa^2}{h_2}\,\frac{ms'}{Mh_2} + \frac{m''d^2}{Mh_2}$$

If we put $h_1h_2 = \kappa^2$ as an approximation in the coefficients of the small terms containing $\frac{m}{M}$ the computed time T is given by

$$\frac{gT^2}{4\pi^2} = \frac{g}{4\pi^2}\left(\frac{h_1T_1^2 - h_2T_2^2}{h_1 - h_2}\right) = h_1 + h_2 + \frac{h_1+h_2}{h_1-h_2}\cdot\frac{ms - ms'}{M} + \frac{(m'-m'')d^2}{M(h_1-h_2)}$$

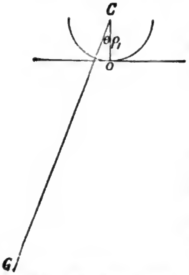

FIG. 2.—Effect of cylindrical Form of Knife Edge.

But if we make the external form of the pendulum symmetrical about its middle point, so that the two knife edges are equidistant from the centre of figure, then $s = s'$ and $m' = m''$

$$\text{and } \frac{g}{4\pi^2}T^2 = h_1 + h_2$$

Then the air effect is eliminated in the computed time. It is necessary here that the barometer and thermometer should give the same readings in each observation; if not, corrections must be made; but, as they will be very small, an exact knowledge of their value is unnecessary.

In investigating the effect of the cylindrical form of the knife edges we shall for simplicity suppose them each to have constant curvature, the radius of the erect one being ρ_1, that of the inverted one ρ_2. If C, Fig. 2, is the centre of curvature of the knife edge, O the point of contact, G the centre of gravity, then $CG = h_1 + \rho_1$ and the work done is the same as if G were moved in a circle of radius $h_1 + \rho_1$, since the horizontal travel of C does not affect the amount of work. The instantaneous centre of motion is the point of contact O. The kinetic energy is therefore

$$M(\kappa^2 + OG^2)\frac{\dot{\theta}^2}{2}$$

But

$$\begin{aligned} OG^2 &= OC^2 + CG^2 - 2\,OC\cdot CG\cos\theta \\ &= \rho_1^2 + (\rho_1 + h_1)^2 - 2\rho_1(\rho_1 + h_1)\left(1 - \frac{\theta^2}{2}\right) \text{ approximately} \\ &= (\rho_1 + h_1 - \rho_1)^2 + \rho_1(\rho_1 + h_1)\theta^2 \\ &= h_1^2 \text{ neglecting } \rho_1 h_1 \theta^2 \text{ and smaller quantities.} \end{aligned}$$

Then the kinetic energy is $M(h_1^2 + \kappa^2)\frac{\dot{\theta}^2}{2}$.

The work done from the lowest point is

$$Mg(h_1 + \rho_1)(1 - \cos\theta) = Mg(h_1 + \rho_1)\frac{\theta^2}{2}$$

Hence the erect time is given by *

$$\frac{gT_1^2}{4\pi^2} = \frac{h_1^2 + \kappa^2}{h_1 + \rho_1} = \frac{h_1^2 + \kappa^2}{h_1}\left(1 - \frac{\rho_1}{h_1}\right) \text{ and}$$

the inverted time is given by

$$\frac{gT_2^2}{4\pi^2} = \frac{h_2^2 + \kappa^2}{h_2}\left(1 - \frac{\rho_2}{h_2}\right)$$

In the computed time we may put $\kappa^2 = h_1 h_2$ in the coefficient of the small quantities ρ_1 and ρ_2, and therefore

$$\frac{gT^2}{4\pi^2} = \frac{g}{4\pi^2}\left(\frac{h_1 T_1^2 - h_2 T_2^2}{h_1 - h_2}\right) = h_1 + h_2 + \frac{h_1 + h_2}{h_1 - h_2}(\rho_2 - \rho_1)$$

Now interchange the knife edges. Assuming that no alteration is made except in the interchange of ρ_1 and ρ_2, the computed time T′ is given by

$$\frac{gT'^2}{4\pi^2} = h_1 + h_2 + \frac{h_1 + h_2}{h_1 - h_2}(\rho_1 - \rho_2);$$

adding the two last equations together and dividing by 2,

$$\frac{g}{4\pi^2}\cdot\frac{T^2 + T'^2}{2} = h_1 + h_2$$

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* If in simple harmonic motion the kinetic energy at any point is $\frac{1}{2}a\dot{\theta}^2$ and the work from the centre of swing is $\frac{1}{2}b\theta^2$, then the periodic time is easily seen to be $2\pi\sqrt{\frac{a}{b}}$

Repsold's Pendulum.—Bessel did not himself construct a pendulum to fulfil these conditions, but, after his death, Repsold in 1860 devised a form with interchangeable knife edges and of symmetrical form now known as Repsold's Reversible Pendulum (Fig. 3), in which he carried out Bessel's suggestions. The stand for the instrument was, perhaps fortunately, far from sufficiently firm, for as the pendulum swung to and fro the stand swung with it. Attention was directed to the investigation of the source of error. Its existence was already known, but its magnitude was not suspected till Peirce and others showed how seriously it might affect the time.

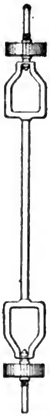

FIG. 3.—Repsold's Reversible Pendulum. The Russian Pendulum used in the Indian Survey.

Yielding of the Support.—The centre of gravity moves as if all the forces acted on the whole mass collected there, so that if we find the mass acceleration of the centre of gravity, and subtract the weight, Mg, we have the force due to the support. Reversing, we have the force on the support.

The acceleration of the centre of gravity is $h_1\ddot{\theta}$ along the arc and $h_1\dot{\theta}^2$ towards the point of support. Resolving these horizontally and vertically,

horizontal acceleration

$$= h_1\ddot{\theta}\cos\theta - h_1\dot{\theta}^2\sin\theta = h_1\ddot{\theta} \text{ approximately;}$$

vertical acceleration

$$= h_1\ddot{\theta}\sin\theta + h_1\dot{\theta}^2\cos\theta = h_1\ddot{\theta}\theta + h_1\dot{\theta}^2 \text{ approximately;}$$

$$\text{but } \ddot{\theta} = -\frac{gh_1\theta}{h_1^2+\kappa^2}$$

Then the horizontal force on the stand is $\mathrm{M}g\dfrac{h_1^2}{h_1^2+\kappa^2}\theta$

$$= \mathrm{M}g\frac{h_1\theta}{h_1+h_2} \text{ since } \kappa^2 = h_1h_2$$

If a is the amplitude of θ, then $\dot{\theta}^2 = \dfrac{gh_1}{h_1^2+\kappa^2}(a^2-\theta^2)$

and the vertical force upwards, on the pendulum

$$= \mathrm{M}g\left\{-\frac{h_1^2\theta^2}{h_1^2+\kappa^2} + \frac{h_1^2}{h_1^2+\kappa^2}(a^2-\theta^2)\right\} - \mathrm{M}g$$

Now in finding the yielding of the stand we only want the varying part of this. Reversing it, the variation in the force on the stand

$$= 2\mathrm{M}g\frac{h_1^2}{h_1^2+\kappa^2}\theta^2 = 2\mathrm{M}g\frac{h_1}{h_1+h_2}\theta^2$$

which is of the second order in θ, and it can be shown that the effect on the time of swing is negligible in comparison with that of the horizontal yielding.

Let the yielding to a horizontal force be e per dyne. Let OC (Fig. 4) be the vertical position, AG the position when displaced through angle θ.

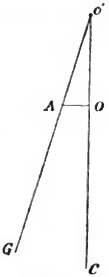

Fig. 4.—Yielding of the Support.

Then the yielding OA $= e\frac{Mh_1}{h_1+h_2}g\theta$

produce GA to O′, then $OO' = OA/\theta = e\frac{Mh_1}{h_1+h_2}g = d_1$ say,

or the instantaneous centre is raised d_1 above O, and the centre of gravity is moving in a circle of radius

$$h_1 + d_1$$

Let the instantaneous centre be raised $d_2 = e\frac{Mh_2}{h_1+h_2}g$ when the pendulum is inverted.

Hence the erect time is given by

$$\frac{gT_1^2}{4\pi^2} = \frac{(h_1+d_1)^2+\kappa^2}{h_1+d_1} = h_1 + d_1 + \frac{\kappa^2}{h_1+d_1}$$

the inverted time by

$$\frac{gT_2^2}{4\pi^2} = \frac{(h_2+d_2)^2+\kappa^2}{h_2+d_2} = h_2 + d_2 + \frac{\kappa^2}{h_2+d_2}$$

and the computed time by $\frac{gT^2}{4\pi^2} = h_1 + h_2 + eMg$, since $h_1 d_2 = h_2 d_1$.

We see that eMg is the horizontal displacement of the support due to the weight of the pendulum applied horizontally.

Defforges' Pendulums.—Starting from this point, Commandant Defforges has introduced a new plan to eliminate the effect of yielding, using two *convertible* pendulums of the Repsold type, of equal weight, of different lengths, and with a single pair of knife edges, which can be transferred from one to the other. The ratio of $h_1 : h_2$ is made the same for each.

Let the radii of curvature of the knife edges be denoted by $\rho_1\ \rho_2$, let $h_1 + h_2 = l_1$ refer to the first pendulum, $h'_1 + h'_2 = l_2$ refer to the second.

The effect of yielding is the same for each, increasing the length by δ.

Let T T′ be their computed times,

then $$\frac{gT^2}{4\pi^2} = l_1 + \delta + \frac{h_1+h_2}{h_1-h_2}(\rho_2-\rho_1)$$

and $$\frac{gT'^2}{4\pi^2} = l_2 + \delta + \frac{h_1'+h_2'}{h_1'-h_2'}(\rho_2-\rho_1)$$

and $\frac{g}{4\pi^2}(T^2 - T'^2) = l_1 - l_2 + (\rho_2 - \rho_1)\left(\frac{l_1}{h_1 - h_2} - \frac{l_2}{h_1' - h_2'}\right)$,

since $\frac{h_1}{h_2} = \frac{h_1'}{h_2'}$ the co-efficient of $\rho_2 - \rho_1$ disappears, and it is not necessary to interchange the knife edges on the same pendulum. Hence the pendulums are convertible, and we have

$$\frac{g}{4\pi^2}(T^2 - T'^2) = l_1 - l_2$$

The United States Coast and Geodetic Survey have recently constructed a pendulum in which the planes are on the pendulum and the knife edges on the support. The one disadvantage is the difficulty of so suspending the pendulum that the same part of the plane is always on the knife edge, but against this is to be set the probable greater accuracy of measurement of $h_1 + h_2$ and the freedom from the necessity of interchange of knife edge. Further, should a knife edge be damaged it can be reground without affecting the pendulum, whereas in the ordinary construction regrinding really alters the pendulum, which practically becomes a different instrument.

Variation of Gravity over the Surface of the Earth.

Richer.—The earliest observation showing that gravity changes with change of place was made by Richer, at the request of the French Academy of Sciences, in 1672. He observed the length of the seconds pendulum at Cayenne, and returning to Paris found that the same pendulum must there be lengthened 1¼ Paris lines, 12 to the inch.

Newton's Theory.—This observation waited no long time for an explanation. Newton took up the subject in the *Principia* (Book III., Props. 18–20) and, regarding gravity as a terrestrial example of universal gravitation, he connected the variation with the form of the earth. He showed first that if the earth is taken as a homogeneous mutually gravitating fluid globe, its rotation will necessarily bring about a bulging at the Equator, for some of the weight of the equatorial portion will be occupied in keeping it moving in its daily circle while the polar part has but little of such motion. A column, therefore, from the centre to the surface must be longer at the Equator than at the Pole in order that the two columns shall produce equal pressures at the centre. Assuming the form to be spheroidal, the attraction will be different at equal distances along the polar and equatorial radii. Taking into account both the variation in attraction and the centrifugal action ($\frac{1}{289}$ of gravity at the Equator), Newton calculated the ratio of the axes of the spheroid. Though his method is open to criticism, his result from the data used is perfectly correct, viz., that the axes are as 230 : 229. Taking a lately measured value of 1° of latitude, he found thence the radii, and determined their difference at 17·1 miles. He then found how gravity should vary over such a spheroid, taking centrifugal action into account, and prepared a table of the lengths of 1° of latitude and of the seconds pendulum for every 5° of latitude from the Equator to the Pole. From his table the pendulum length at Cayenne, in latitude 4° 55′, should be 1 line less than at Paris in latitude 48° 50′. He assigns part of the difference of this from the diminution of $1\frac{1}{4}$ lines observed by Richer to expansion of the scale with higher temperature near the Equator.

The Swedish and Peruvian Expeditions.—Newton's theory of the figure of the earth as depending on gravitation and rotation led early in the eighteenth century to measurements of a degree of latitude in Peru and in Sweden. If the earth were truly spheroidal, and if the plumb-line were everywhere perpendicular to the surface, two such measurements would suffice to give the axes a and b, inasmuch as length of arc of 1°

$$= b\left(1 - \epsilon + 3\epsilon \sin\frac{\lambda + \lambda'}{2}\right)3600 \sin 1'' \text{ where } \epsilon = \frac{a - b}{a} = \text{the ellipticity and } \lambda\ \lambda'$$

are the latitudes at the beginning and end of the arc.*

We know now that through local variations in gravity the plumb-line is not perpendicular to a true spheroid, but that there are humps and hollows in the surface, and many measurements at different parts of the earth are needed to eliminate the local variations and find the axes of the spheroid most nearly coinciding with the real surface. But the Swedish and Peruvian expeditions clearly proved the increase of length of a degree in northerly regions, and so proved the flattening at the Poles. These expeditions have another interest for us here in that pendulum observations were made. Thus Maupertuis, in the northern expedition, found that a certain pendulum clock gained 59·1 seconds per day in Sweden on its rate in Paris, while Bouguer and La Condamine, in the Peruvian expedition, found that at the Equator at sea-level the seconds pendulum was 1·26 Paris lines shorter than at Paris. Bouguer's work, to which we have already referred, was especially important in that he determined the length of the seconds pendulum at three elevations: (1) At Quito, which may be regarded as a tableland, the station being 1466 toises† above sea-level; (2) on the summit of Pichincha, a mountain rising above Quito to a height of 2434 toises above sea-level; and (3) on the Island of Inca, on the river Esmeralda, not more than thirty or forty toises above sea-level. The Equator runs between Quito and the third station, and they are only a few miles from it. In space free from matter rising above sea-level gravity might be expected to decrease according to the inverse square law starting from the earth's centre, so that if h is the height above sea-level and r is

Station.	Above Sea-level in Toises.	Observed Seconds Pendulum in Lines.	Correction for Temperature.	Correction for Buoyancy.	Corrected Seconds Pendulum.	Fraction less than at Sea-level.	Fraction given by Inverse Square Law $2h/r$.
Pichincha .	2434	438·70	− ·05	+ ·04	438·69	$\frac{1}{845}$	$\frac{1}{671}$
Quito . .	1466	438·83	—	+ ·05	438·88	$\frac{1}{1331}$	$\frac{1}{1118}$
Isle of Inca	—	439·07	+ ·075	+ ·06	439·21	—	—

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* Airy, "Figure of Earth," *Encyc. Met.*, p. 192.

† The toise is 6 Paris feet, or 6·395 English feet.

the earth's radius, the decrease should be $2h/r$ of the original value. In the table on p. 21, Bouguer's results are given. In the last column but one is the decrease observed at the upper stations, and in the last column the decrease calculated by $2h/r$.

It will be seen that gravity decreased more slowly than by the inverse square law. Centrifugal force would act in a contrary way, though, as Bouguer showed, by a negligible amount. The excess of gravity, as observed, above its value in a free space must therefore be assigned to the attraction of the matter above the sea-level. Bouguer obtained for the value of gravity g_h on a plateau of height h, as compared with its value at sea-level g_s

$$g_h = g_s\left(1 - \frac{2h}{r} + \frac{3}{2}\frac{h}{r}\frac{\delta}{\Delta}\right),$$

where δ is the density of the plateau and Δ the density of the earth.

This formula, now known as Bouguer's Rule, seems to have dropped out of sight till it was again obtained by Young in 1819, but on its revival it was generally employed to reduce the observed value at a station to the sea-level value in the same latitude.

Putting it in the form $\dfrac{g_s - g_h}{g_s} = \dfrac{2h}{r}\left(1 - \dfrac{3}{4}\dfrac{\delta}{\Delta}\right)$

and using the values at Quito and sea-level, $\Delta = \dfrac{3993}{850}\delta$

Bouguer remarked that this result sufficed to show that the density of the earth was greater than that of the Cordilleras, and consequently that the earth was neither hollow nor full of water, as some physicists had maintained. We now know that the value of Δ so obtained is far too great, and shall see later what is the probable explanation.

Clairaut's Theorem.—In 1743 Clairaut published his great treatise, *Théorie de la Figure de la Terre*, which put the investigation of the figure of the earth on lines which have ever since been followed. In this work he takes the surface of the earth as a spheroid of equilibrium—*i.e.*, such that a layer of water would spread all over it, and assumes that the internal density varies so that layers of equal density are concentric co-axial spheroids. Denoting gravity at the Equator, Pole, and latitude λ, by g_e, g_p, g_λ respectively, and putting m = centrifugal force at Equator $/g_e$ and ϵ = ellipticity = difference of equatorial and polar radii / equatorial radius, he shows (1) that

$$g_\lambda = g_e(1 + n\sin^2\lambda) \qquad (1)$$

where n is a constant: (2) that

$$\frac{g_p - g_e}{g_e} + \epsilon = \frac{5}{2}m. \qquad (2)$$

From (1) and (2) we get

$$g_\lambda = g_e\left\{1 + \left(\frac{5}{2}m - \epsilon\right)\sin^2\lambda\right\},$$

a result known as Clairaut's Theorem.

Laplace showed that the surfaces of equal density might have any nearly spherical form, and Stokes (*Math. Phys. Papers*, vol. ii. p. 104), going further, showed that it is unnecessary to assume any law of density so long as the external surface is a spheroid of equilibrium, for the theorem still remains true.

From Clairaut's Theorem it follows that, if the earth is an oblate spheroid, its ellipticity can be determined from pendulum experiments on the variation of gravity without a knowledge of its absolute value, except in so far as it is involved in m. And if the theorem were exactly true, two relative determinations at stations in widely different latitudes should suffice. But here again, as with arc measurements, local variations interfere, and many determinations must be made at widely scattered stations to eliminate their effect.

Kater and Sabine. Invariable Pendulums.—During the last half of the eighteenth century much pendulum work was carried on, but hardly with sufficient accuracy to make the results of value now, and we may consider that modern research begins with Kater, who constructed a number of "invariable pendulums," nearly beating seconds, and in shape much like his convertible pendulum without the reverse knife edge. The principle of "invariable pendulum" work consists in using the same pendulum at different stations, determining its time of vibration at each, and correcting for temperature, air effect, and height above sea-level. The relative values of gravity are thus known, or the equivalent, the relative lengths of the seconds pendulum, without measuring the length or knowing the moment of inertia of the pendulum. Kater himself determined the length of the seconds pendulum at stations scattered over the British Islands, and Sabine, between 1820 and 1825, carried out observations at stations ranging from the West Indies to Greenland and Spitzbergen. About the same time Freycinet and Duperry made an extensive series ranging far into the Southern Hemisphere, and other observers contributed observations. Now, though different pendulums were used, these series overlapped and could be connected together by the observations at common stations; and Airy in 1830 (*Encyc. Met.*, "Figure of the Earth") deduced a value of the ellipticity of about $\frac{1}{283}$.

Breaking down of Bouguer's Rule.—Subsequent work brought into ever-increasing prominence the local divergences from Clairaut's formula, and it gradually became evident that on continents and on high ground the value of gravity was always less than would be expected from Clairaut's formula when corrected by Bouguer's rule, while at the sea coast and on oceanic islands it was greater.

Indian Survey.—Thus, in the splendid series of pendulum experiments carried out in connection with the Indian Trigonometrical Survey between 1865 and 1875 (*G. T. Survey of India*, vol. v.) the variations were very marked. In these experiments, invariable pendulums, Kater's convertible and Repsold's reversible pendulum were all used, and observations were made by Basevi and Heaviside from Moré, on the Himalayas, at a height of 15,427 feet, down to the sea-level. The series was connected with others by swinging the pendulums at Kew before their transmission to India, and very great precautions were taken to correct for temperature, and the air effect was eliminated by swinging in a vacuum. At Moré the defect of gravity was very marked.

Airy's "Hydrostatic" Theory. Faye's Rule.—Airy (*Phil. Trans.*, 1855, p. 101) had already suggested that elevated masses are really buoyed up by matter at their base lighter than the average; that in fact they float on the liquid or more probably viscous solid interior very much as icebergs float on the sea. If the high ground is in equilibrium, neither rising nor falling, we may perhaps regard the total quantity of matter underneath a station as being equal to that at a station at sea-level in the same latitude. This hydrostatic theory has led Faye to suggest that the term $\frac{3}{2}\frac{\delta}{\Delta}\frac{h}{r}$ in Bouguer's rule should be replaced by a term only taking into account the attraction of the excess of matter under the station above the average level of the near neighbourhood, a suggestion embodied in Faye's rule.

Recent work by the American Survey (*Amer. Journ. Science*, March 1896, G. R. Putnam) has shown that on the American continent Faye's rule gives results decidedly more consistent than those obtained from Bouguer's rule.

By a consideration of the results obtained up to 1880 by the pendulum, Clarke (*Geodesy*, p. 350) gives as the value of the ellipticity $\epsilon = \frac{1}{292 \cdot 2 \pm 1 \cdot 5}$, a value almost coinciding with that obtained from measurements of degrees of latitude. Helmert, in 1884, gave as the result of pendulum work $\frac{1}{299 \cdot 3}$, and we may now be sure that the value differs very little from $\frac{1}{300}$.

Helmert (*Theorieen der höheren Geodäsie*, Bd. II. p. 241) also gives as the value of g in any latitude λ,

$$g_\lambda = 978 \cdot 00(1 + 0 \cdot 005310 \sin^2 \lambda)$$

and this may be taken as representing the best results up to the present.

Von Sterneck's Half-second Pendulums.—The labour of the determination of minute local variations in gravity was much lessened by the introduction by von Sterneck, about 1880, of half-second invariable pendulums, and his improved methods of observation have greatly increased the accuracy of relative determinations at stations connected by telegraph.

With half the time of swing the apparatus has only one-fourth the linear dimensions, and it can be made at once more steady and more portable. The size of the pendulum being thus reduced—it is about 10 inches long—it can without much trouble be placed in a chamber which can be exhausted and which can be maintained at any desired temperature. Each pendulum can therefore be made to give its own temperature and air corrections by preliminary observations. The form of the pendulum is shown in Fig. 5. The chief improvements in the mode of observation introduced by von Sterneck consist, 1st, in the simultaneous comparison with the same clock of the swinging of two pendulums at two stations at which gravity is to be compared. For this purpose the two stations are connected by an electric circuit containing a half-seconds "break circuit" chronometer, which sends a signal through each station every half-second, and thus clock-rates are of little importance. And, 2nd, the method of observing the coincidences of the pendulum with the chronometer signals. In the final form this consists in attaching a small mirror on the pendulum knife edge (not shown in Fig. 5, which represents an earlier form) perpendicular to the plane of vibration of the pendulum, and placing a fixed mirror close to the other and parallel to it when the pendulum is at rest.

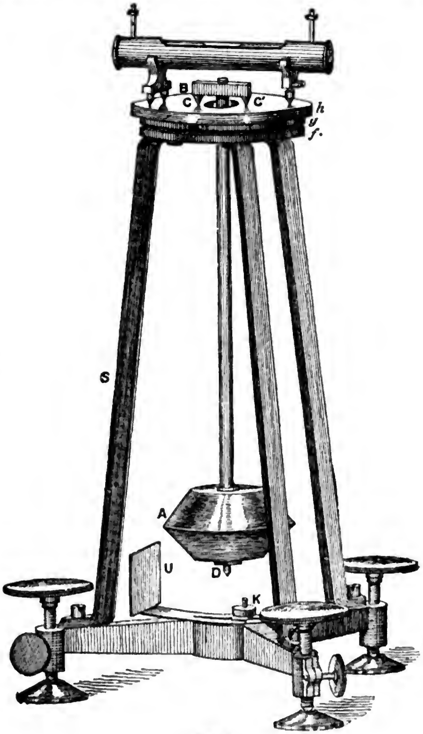

Fig. 5.

The chronometer signals work a relay, giving a horizontal spark, and this is reflected into a telescope from both mirrors. When the pendulum is at rest the image of the spark in both mirrors appears on the horizontal cross-wire, and when the pendulum is vibrating a coincidence occurs when the two images are in this position. The method admits of exceedingly accurate determination. We shall see later how von Sterneck used the method in gravitation experiments. Here it is sufficient to say that he has used it in many local determinations of gravity, and that his pendulums have been used without the simultaneous method for determinations at various stations in both hemispheres. The American Geodetic Survey has adopted very similar apparatus and methods, and it appears probable that we shall soon have a knowledge of the variation of gravity over the surface of the earth of a far more detailed and accurate kind than could possibly be obtained by the older methods.

Differential Gravity Meters.—Before invariable pendulums were brought to their present accuracy and portability, there was some hope that for relative determinations the pendulum might be superseded by a statical measurer of gravity which would do away with the need for time measurements. Such an instrument must essentially consist of a mass supported by a spring, and the variation in gravity must be shown by the alteration in the spring due to the alteration in the pull of the earth on the mass. The earlier instruments devised for the purpose need not be described, for they were quite incapable of the accuracy attained by invariable pendulums. The first instrument which promised any real success was devised by von Sterneck, and is termed by him the Barymeter (*Mittheilungen des K. K. Militar-Geog. Inst.*, Wien, v. 1885).

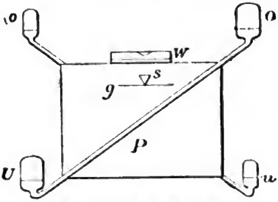

FIG. 6.—Von Sterneck's Barymeter.

Von Sterneck's Barymeter.—A brass plate P (Fig. 6), 30 cm. × 20 cm., is balanced on a knife edge, *s*. Along a diagonal is a glass tube terminating in bulbs O and U, 5 cm. × 6 cm., so that in the equilibrium position O is about 25 cm. above U. The tube and about $\frac{1}{5}$ of each bulb is filled with mercury, and above the mercury is nitrogen. The apparatus is adjusted so that at 0° C. and for a certain value of gravity the edges of the brass plate are horizontal and vertical, a level W showing when this position is attained. If now gravity were to increase, the weight of the mercury would be greater, and it would tend to flow from O and compress the gas in U. Thus the balance would tilt over to the left, and the tilting still further increasing the pressure on U, the flow downwards is increased. The instrument can thus be made of any desired sensitiveness, and its deflections can be read by scale and measured in the usual way. To compensate for changes of temperature, a second tube terminating in smaller bulbs *o* and *u*, each about 6 cm. × 3 cm., is fixed along the other diagonal. This contains some mercury, but above the mercury in *u* is alcohol, and only *o* contains nitrogen. If the temperature rises the mercury becomes less dense, and on this account it is driven from U to O in the larger tube, but still more is it driven in this direction from the fact that the increase of pressure of the gas in U is greater than in O. Meanwhile, the alcohol in *u* expanding, drives the mercury in the smaller tube into *o*, and by suitable adjustments of volume the two can be made to balance sufficiently for such small temperature variations as will arise when the whole is placed in a box surrounded with melting ice, and it is thus that the instrument is used. With this instrument von Sterneck could detect the change in gravity in going from the cellar of a building to a height of 25 metres.

Threlfall and Pollock's Quartz-thread Gravity Balance.—In the *Phil. Trans.*, A. 193, 1899, p. 215, Threlfall and Pollock describe an instrument for measuring variations in gravity statically which is both accurate and portable.

The essential features of the instrument are represented in Fig. 7.

A and B are two metal rods which can slide along their common axis. C is a coach-spring attached to A. H I is a quartz thread 30·5 cm. long and ·0038 cm. in diameter stretched horizontally between B and C. D is a piece of gilded brass wire soldered to the quartz thread. Its weight is ·018 gm., its length 5·3 cm., and its centre of gravity is a little to one side of the quartz thread. Its weight therefore tends to pull it into the vertical position and twist the quartz. But such a twist can be put on the quartz thread by rotating the arm G, which carries a vernier, that D is brought into the horizontal position. For this about three whole turns are required. The end of D when in the horizontal position is on the cross-wire of the horizontal microscope E. The horizontal position of the brass wire is only just stable. If it be twisted a few degrees more the point of instability is reached and the wire tends to continue moving round, and would do so but for an arrester. The mode of using the instrument consists in determining the twist put on the quartz thread by the arm G to bring it into the horizontal position. If gravity increases, the moment of the weight of D increases and a greater twist is required. To calibrate the instrument the change in reading of the vernier on G is observed in passing from one station to another, at both of which g is known—the two stations selected being Sydney and Melbourne. Of course, temperature corrections are necessary both on account of the change in length of D and the change in rigidity of the quartz. Preliminary determinations of these were made at one station. For the details of the instrument and the mode of using it we refer the reader to the original account. It suffices here to say that it has given very fairly consistent results at stations wide apart and that it promises to rival the invariable pendulum.

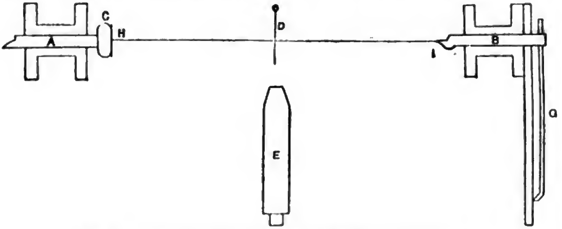

FIG. 7.—Threlfall and Pollock's Quartz-thread Gravity Balance.

CHAPTER III.

GRAVITATION.

CONTENTS.*—The Law of Gravitation—The Gravitation Constant and the Mean Density of the Earth.

THE full statement of Newton's Law of Gravitation is that any particle of mass M_1 attracts any other particle of mass M_2 distant d from it with a force in the line joining them proportional to M_1M_2/d^2. The evidence for the law may be briefly summed up as follows:

Starting with any single planet—say the earth—and referring its position to a system, fixed relatively to the sun and the distant stars, direct astronomical observation shows that it may be described with a close approximation to the truth, as moving in an ellipse with the sun in one focus, at such speed that the line from the centre of the sun to the centre of the planet sweeps out equal areas in equal times. This implies, as Newton showed, that the acceleration of the planet is towards the sun and inversely as the square of its distance from that body.

Now, comparing the different planets, observation shows that (length of year)^2/(mean distance)^3 is the same for each, and from this it follows that the constant of acceleration is the same for all, or that at the unit distance from the sun they would all have the same acceleration if the law holding for each in its own orbit held for it at all distances.

So far this is mere time-geometry, or a description of position and rate of change of position, and we might have other equally true, if less convenient, modes of description referred to other standards, such as the epicyclic geocentric mode of the ancients, or the practical mode in common use in which the co-ordinates of a planet are measured with regard to some observatory, its meridian, and horizon.

But if we regard the accelerations as indicating forces, the different methods of description are no longer equivalent. We must select that which gives a system of forces most consistent in itself and most in accord with our terrestrial experience. Here the heliocentric method, with the modification described hereafter, is immensely superior to any other, and, adopting it, we must suppose that the accelerations of the planets indicate forces towards the sun, and since the constant of acceleration is the same for all, that the forces on equal masses are inversely as their distances squared from the sun, whatever planets the masses belong to. In other words, the sun has no favourite among its attendants, but pulls on each pound of each according to the same rule.

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* This chapter is largely taken from *The Mean Density of the Earth*, and papers communicated to the Royal Institution and the Birmingham Natural History and Philosophical Society, by J. H. Poynting.

But the assumption that the accelerations indicate forces of the kind we experience on the earth, carries with it the supposition of equality of action and reaction, and so we conclude that each planet reacts on the sun with a force equal and opposite to that exerted by the sun on the planet. Hence, each acts with a force proportional to its own mass, and inversely as the square of its distance away. If we suppose that there is nothing special in the attraction of the sun beyond great magnitude corresponding to great mass, we must conclude that the sun also acts with a force proportional to its mass. But we have just shown that the force is proportional to the mass acted on. Hence, we have the force on any planet proportional to mass of sun × mass of planet / (distance apart)^2.

Now, turning to any of the smaller systems consisting of a primary and its satellites, the shape of orbit and the motion of the satellites agree with the supposition that the primary is acting with a force according to the inverse square law. It is important for our special problem to note here that in the case of the earth we must include in the term "satellite" any body at its surface which can be weighed or moved.

We are therefore led to conclude that the law is general, or that if we have any two bodies, of masses M_1 and M_2, at d distance apart, the force on either is

$$\frac{GM_1M_2}{d^2}$$

where G is a constant—the constant of gravitation.

The acceleration of one of them, say M_2, towards the other is $\frac{GM_1}{d^2}$

If this conclusion is accepted, we can at once determine the masses of the various primaries in terms of that of the sun for—

$$\text{acceleration of satellite towards primary} = G\frac{\text{Mass of primary}}{\text{distance of satellite}^2}$$

$$\text{and acceleration of primary towards sun} = G\frac{\text{Mass of sun}}{\text{distance of primary}^2}$$

By division G is eliminated, and we obtain the ratio of the masses in terms of quantities which may be measured by observation.

As an illustration, let us make a rough determination of the mass of the sun in terms of the mass of the earth.

We may take the acceleration of the moon to the earth as approximately $\omega_M{}^2 \times d_M$, where ω_M is the angular velocity of the moon and d_M its distance from the earth, and the acceleration of the latter to the sun as $\omega_E{}^2 \times d_E$ where ω_E is the angular velocity of the earth, and d_E its distance from the sun. Let the mass of the sun be S and that of the earth be E.

$$\text{then} \quad \frac{\text{Acceleration of Moon}}{\text{Acceleration of Earth}} = \frac{\omega_M{}^2 \times d_M}{\omega_E{}^2 \times d_E} = \frac{E \times d_E{}^2}{S \times d_M{}^2},$$

$$\text{whence} \quad \frac{S}{E} = \frac{\omega_E{}^2 d_E{}^3}{\omega_M{}^2 d_M{}^3} = \left(\frac{27}{365}\right)^2 \left(\frac{92000000}{240000}\right)^3 = 300000$$

A confirmation of the generality of the law is obtained from the perturbations of the planets from the elliptic orbits which we have for simplicity supposed them to describe.

These perturbations, in any one planet, can at least approximately be analysed into separate disturbances, each due to one of its fellow planets, acting with a force inversely as the square of its distance away, and if we assume this force proportional to the mass of the disturber we obtain another measure for this mass in terms of that of the sun.

The concordance of the two methods is as complete as we could expect.

The determination of the masses of the different members of our system in terms of that of the sun enables us to choose a still more satisfactory origin for our system of reference than the centre of the sun—viz., the centre of mass of the whole system. The change is small, but without it we could not account for all the motions merely by a set of inverse square forces in which action and reaction were equal and opposite.

We have for simplicity considered the sun and planets as without appreciable dimensions as compared with their distances apart. But measurement shows that they are all approximately spheres, and the attraction on a sphere with density varying only with the distance from the centre—*i.e.*, consisting of homogeneous concentric shells, if it is considered as the resultant of the attractions on the separate particles, all according to the same inverse square law, is the same as that on the whole mass collected at the centre of the sphere. Further, if the attraction is due, not to the attracting body as a whole but to its separate parts, each acting, as it were, independently and according to the same law, then an attracting sphere acts as if it were all concentrated at its centre. Since the planets, with a close approximation, behave as if they were merely concentrated masses at their centres, and since the deviations from this behaviour, such as the earth's precession, can all be accounted for by their departure from sphericity, we have strong presumption that the attraction is really the resultant of all the attractions, each element m_1 of one body acting on each element m_2 of the other with force Gm_1m_2/d^2.

Astronomical observation enables us, then, to compare the masses of the various members of the solar system with each other, and, by taking into account the sizes of the planets, to make a table of specific gravities, choosing any one as the standard substance. Thus, if we take the earth as standard, the mean specific gravity of the sun is about 0·25, that of Mercury about 1·25, that of Venus and Mars about 0·9, and so on.

But this does not give us any idea of the specific gravity in terms of known terrestrial substances or any idea of the masses in terms of the terrestrial standards, the kilogramme or the pound. It is true that Newton, with little more than the astronomical data at his command, made a celebrated guess on the specific gravity of the earth in terms of water, which runs thus in Motte's translation of the *Principia* (vol. ii. p. 230, ed. 1729, Book III., Prop. 10): "But that our globe of earth is of greater density than it would be if the whole consisted of water only, I thus make out. If the whole consisted of water only, whatever was of less density than water, because of its less specific gravity, would emerge and float above. And upon this account, if a globe of terrestrial matter, covered on all sides with water, was less dense than water, it would emerge somewhere: and the subsiding water falling back, would be gathered to the opposite side. And such is the condition of our earth, which, in great measure, is covered with seas. The earth, if it was not for its greater density, would emerge from the seas, and according to its degree of levity, would be raised more or less above their surface, the water and the seas flowing backwards to the opposite side. By the same argument, the spots of the sun which float upon the lucid matter thereof, are lighter than that matter. And however the Planets have been form'd while they were yet in fluid masses, all the heavier matter subsided to the centre. Since, therefore, the common matter of our earth on the surface thereof, is about twice as heavy as water, and a little lower, in mines is found about three or four, or even five times more heavy; it is probable that the quantity of the whole matter of the earth may be five or six times greater than if it consisted all of water, especially since I have before shewed that the earth is about four times more dense than Jupiter."

It is not a little remarkable that Newton hit upon the limits between which the values found by subsequent researches have nearly all lain.

In order, then, to complete the expression of the law of gravitation we must connect the celestial with the terrestrial scale of densities. In fact, we must do for the masses of the solar system that which we do for their distances in the determination of the solar parallax, though we cannot proceed quite so directly in the former case as in the latter in connecting the celestial and terrestrial measures. If we could measure the acceleration, say, of the moon, due to any terrestrial body of known shape and density—if, for instance, we knew the form and extent of our tidal-wave and its full lunar effect—we could at once find the mass of the earth in terms of that of the wave, or its density as compared with sea-water.

But at present this cannot be done with any approach to accuracy, and the only method of solving the problem consists in finding the attraction between two bodies on the earth of known masses a known distance apart, and comparing this with the attraction of the earth on a known mass at its surface instead of its attraction as a heavenly body. Since the law of attraction is by observation the same at the surface of the earth and at a distance, we can thus find the mass of the earth in terms of either of these known masses.

To take an illustration from an experiment hereafter described, let us suppose that a spherical mass of 20 kilos. is attracted by another spherical mass of 150 kilos. when the centres are 30 cm. apart with a force equal to the weight of $\frac{1}{4}$ mgm. or $\frac{1}{80000000}$ of the weight of the 20 kilos. when the latter is on the surface of the earth and 6×10^8 cm. from its centre, we have:

$$\frac{\text{Mass of Earth}}{(6 \times 10^8)^2} : \frac{150000}{30^2} = 1 : \frac{1}{80000000}$$

whence mass of earth $= 5 \times 10^{27}$ grammes nearly.

The volume of the earth is about 9×10^{26} c.c., whence the mean density of the earth Δ is about 5·5.

Or, using the experiment to give the constant of attraction, and expressing the masses in grammes, the weight of $\frac{1}{4}$ mgm. or

$$\cdot 00025g = \frac{G \times 150000 \times 20000}{30^2}$$

Whence, if $g = 981$, $G = \frac{981 \times \cdot 00025 \times 30^2}{150000 \times 20000} = \frac{7}{10^8}$ (nearly).

A determination of G completes the expression of the law of gravitation.

This example shows that the two problems, the determination of the gravitation constant G and the determination of the mean density of the earth Δ, are practically one, inasmuch as our knowledge of the dimensions of the earth and the acceleration of gravity g at its surface at once enable us to determine G if we know Δ, or to determine Δ if we know G.

The Methods of Experiment.

These naturally fall into two classes. In the one class some natural mass is selected, either a mountain or part of the earth's crust, and its mass and form are more or less accurately determined by surveys and mineralogical examination. Its attraction on a plumb-bob at one side, or on a pendulum above or below it, is then compared with the attraction of the whole earth on the same body.

In the other, the laboratory class of experiment, a smaller mass, such as may be easily handled, is placed so as to attract some small suspended body, and this attraction is measured. Knowing the attracting and attracted masses, the attraction gives G. Or, comparing the attraction with the attraction of the earth on the same body, we get Δ.

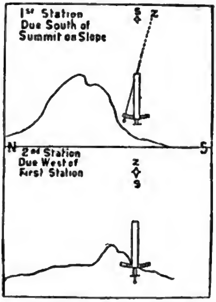

Fig. 8.—Bouguer's Plumb-line Experiment on the Attraction of Chimborazo.

The Experiments of Bouguer in Peru.—The honour of making the first experiments on the attraction of terrestrial masses is to be accorded to Bouguer. He attempted both by the pendulum experiments described in the last chapter, and by plumb-line experiments, to prove the existence of the attraction of mountain masses in the Andes, when engaged in the celebrated measurement of an arc of the meridian in Peru about the year 1740. The pendulum experiments are sufficiently described in the last chapter.

In his plumb-line experiments he attempted to estimate the sideway attraction of Chimborazo, a mountain about 20,000 feet high, on a plumb-line placed at a point on its side. Fig. 8 will show the principle of the method. Suppose that two stations are fixed, one on the side of the mountain due south of the summit, and the other in the same latitude, but some distance westward, away from the influence of the mountain. Suppose that at the second station a star is observed to pass the meridian—we will say, for simplicity, directly overhead, then a plumb-line hung down will be exactly parallel to the observing telescope. At the first station, if the mountain were away, it would also hang down parallel to the telescope when directed to the same star. But the mountain pulls the plumb-line towards it, and changes the overhead point so that the star appears to northward instead of in the zenith. The method simply consists in determining how much the star appears to be shifted to the north. The angle of apparent shift is the ratio of the horizontal pull of the mountain on the plumb-bob to the pull of the earth.

To carry out the experiment, Bouguer fixed the first station on the south slope of Chimborazo, just above the perpetual snow-line, and the second nearly on the same level, several miles to the westward. He describes (*Figure de la Terre*, 7th section) how his expedition reached the first station after a most toilsome journey of ten hours over rocks and snow, and how, when they reached it, they had all the time to fight against the snow, which threatened to bury their tent. Nevertheless, they succeeded in making the necessary observations, and a few days later they were able to move on to the second station. Here they hoped for better things, as they were now below the snow-line. But their difficulties were even greater than before, as now they were exposed to the full force of the wind, which filled their eyes with sand and was continually on the point of blowing away their tent. The cold was intense, and so hindered the working of their instruments that they had to apply fire to the levelling screws before they could turn them. Still they made their observations, and found that the plumb-line was drawn aside about 8 seconds. Had Chimborazo been of the density of the whole earth, Bouguer calculated, from the dimensions and distance of the mountain, that it would have drawn aside the vertical by about twelve times this, so that the earth appeared to be twelve times as dense as the mountain, a result undoubtedly very far wide of the truth. But it is little wonder that under such circumstances the experiment failed to give a good result, and all honour is due to Bouguer for the ingenuity and perseverance which enabled him to obtain any result at all. At least he deserves the credit of first showing that the attraction by mountain masses actually exists, and that the earth, as a whole, is denser than the surface strata. As he remarks, his experiments at any rate proved that the earth was not merely a hollow shell, as some had till then held; nor was it a globe full of water, as others had maintained. He fully recognised that his experiments were mere trials, and hoped that they would be repeated in Europe.

Thirty years later his hope was fulfilled. Maskelyne, then the English Astronomer Royal, brought the subject before the Royal Society in 1772, and obtained the appointment of a committee "to consider of a proper hill whereon to try the experiment, and to prepare everything necessary for carrying the design into execution." Cavendish, who was himself to carry out an earth-weighing experiment some twenty-five years later, was probably a member of the committee, and was certainly deeply interested in the subject, for among his papers have been found calculations with regard to Skiddaw, one of several English hills at first considered. Ultimately, however, the committee decided in favour of Schiehallion, a mountain near L. Rannoch, in Perthshire, 3547 feet high. Here the astronomical part of the experiment was carried out in 1774, and the survey of the district in that and the two following years. The mountain has a short east and west ridge, and slopes down steeply on the north and south, a shape very suitable for the purpose.

Maskelyne, who himself undertook the astronomical work, decided to work in a way very like that followed by Bouguer on Chimborazo, but modified in a manner suggested by him. Two stations were selected, one on the south and the other on the north slope. A small observatory was erected first at the south station, and the angular distance of some stars from the zenith, when they were due south, was most carefully measured. The stars selected all passed nearly overhead, so that the angles measured were very small. The instrument used was the zenith sector, a telescope rotating about a horizontal east and west axis at the object-glass end, and provided with a plumb-line hanging from the axis over a graduated scale at the eyepiece end. This showed how far the telescope was from the vertical.

After about a month's work at this station the observatory was moved to the north station, and again the same stars were observed with the zenith sector. Another month's work completed this part of the experiment. Fig. 9 will show how the observations gave the attraction due to the hill. Let us for the moment leave out of account the curvature of the earth, and suppose it flat. Further, let us suppose that a star is being observed which would be directly overhead if no mountain existed. Then evidently at S. the plumb-line is pulled to the north, and the zenith is shifted to the south. The star therefore appears slightly to the north. At N. there is an opposite effect, for the mountain pulls the plumb-line southwards, and shifts the zenith to the north; and now the star appears slightly to the south. The total shifting of the star is double the deflection of the plumb-line at either station due to the pull of the mountain.

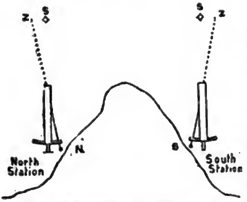

FIG. 9.—Maskelyne's Plumb-line Experiment on Schiehallion.

But the curvature of the earth also deflects the verticals at N. and S., and in the same way, so that the observed shift of the star is partly due to the mountain and partly due to the curvature of the earth. A careful measure was made of the distance between the two stations, and this gave the curvature deflection as about 43″. The observed deflection was about 55″, so that the effect of the mountain, the difference between these, was about 12″.

The next thing was to find the form of the mountain. This was before the days of the Ordnance Survey, so that a complete survey of the district was needed. When this was complete, contour maps were made, giving the volume and distance of every part of the mountain from each station. Hutton was associated with Maskelyne in this part of the work, and he carried out all the calculations based upon it, being much assisted by valuable suggestions from Cavendish.

Now, had the mountain had the same density as the earth, it was calculated from its shape and distance that it should have deflected the plumb-lines towards each other through a total angle of 20·9″, or $1\frac{4}{5}$ times the observed amount. The earth, then, is $1\frac{4}{5}$ times as dense as the mountain. From pieces of the rock of which the mountain is composed, its density was estimated as $2\frac{1}{2}$ times that of water. The earth should have, therefore, density $1\frac{4}{5} \times 2\frac{1}{2}$ or $4\frac{1}{2}$. An estimate of the density of the mountain, based on a survey made thirty years later, brought the result up to 5. All subsequent work has shown that this number is not very far from the truth.

An exactly similar experiment was made eighty years later, on the completion of the Ordnance Survey of the kingdom. Certain anomalies in the direction of the vertical at Edinburgh led Colonel James, the director, to repeat the Schiehallion experiment, using Arthur's Seat as the deflecting mountain. The value obtained for the mean density of the earth was about 5⅓.

Repetitions have also been made of the pendulum method, tried by Bouguer in the Andes.

The first of these was by Carlini, in 1821. He observed the length of a pendulum swinging seconds at the Hospice on Mont Cenis, about 6000 feet above sea-level, and so obtained the value of gravity there. The value due to mere elevation above the sea-level was easily calculated, but the observed value was greater than that calculated by about 1 in 5000. In other words, the pull of the whole earth was 5000 times greater than that of the mountain under the Hospice. Knowing approximately the shape of the mountain, and estimating its density from specimens of the rock, Carlini found the density of the earth to be about 4½ times that of water.

Another experiment of the same kind was made by Mendenhall, in Japan, in 1880. Here he determined the value of gravity on the summit of Fujiyama, a mountain nearly 2½ miles high. He found it greater than the value calculated from the increased distance from the earth's centre by about 1 in 5000, as Carlini had done on Mont Cenis. Fujiyama, though the higher, is more pointed and less dense than Mont Cenis. Mendenhall estimated the mean density of the earth as 5·77.

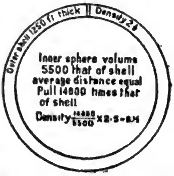
Outer shell 1250 ft thick
Density 2·5
Inner sphere volume 5500 that of shell
average distance equal
Pull 14000 times that of shell
Density 14000/5500 × 2·5 = 6·4

Fig. 10.—Principle of Airy's Harton Pit Experiment.

Airy applied the pendulum to solve the problem in a somewhat different way, using, instead of a mountain, the crust of the earth between the top and the bottom of a mine. His first attempts were made in 1826, at the Dolcoath copper mine, in Cornwall. Here he swung a pendulum first at the surface and then at the bottom of the mine. At the point below we may consider that the weight of the pendulum was due to the pull of the part of the earth within the sphere with radius reaching from the earth's centre to the point (Fig. 10). Knowing the value of gravity below, it was easy to calculate what it would have been at the level of the surface had no outer shell existed, and had the change in value depended merely on the greater distance from the earth's centre. The observed value was greater than this through the pull of the outer shell, and it was hoped that the difference would be measured sufficiently accurately to show how much greater is the mass of the earth than that of the crust. The first attempt was brought to an end by a curious accident. As one of the pendulums used was being raised up the shaft, the box containing it took fire, the rope was burnt, and the pendulum fell to the bottom. Two years later another attempt was made, but this was brought to an end by a fall in the mine, which stopped the pump so that the lower station was flooded.

Many years later, in 1854, the experiment was again undertaken by Airy, this time in the Harton coal-pit, near Sunderland. The method was exactly the same, a pendulum being swung above and below the surface, and the diminution in gravity above carefully determined. The experiment was carried out with the greatest care and in a most thorough way, two pendulums being swung at the same time—one above and one below—the two being interchanged from time to time. Several assistants were occupied in taking the observations, which extended continuously night and day for about three weeks. Now gravity at the surface was greater than it would have been, had no outer shell existed of thickness equal to the depth of the pit, by about 1 in 14,000, so that the pull of the earth was about 14,000 times that of the shell. The density of the shell was determined from specimens of the rocks, and Airy found the density of the earth about 6½.

Some very interesting experiments have since been made in a similar way by Von Sterneck in silver mines in Saxony and Bohemia. Using the invariable pendulums described in the last chapter he obtained different results with different depths of mines, the value of the mean density increasing with the increasing thickness of the shell used. This shows very evidently that there were sources of disturbance vitiating the method. Von Sterneck found, on comparing his observations at the two mines, that the increase in gravity on descending was much more nearly proportional to the rise of temperature than to the depth of descent. This appears to indicate that whatever disturbs the regularity of gravity disturbs also the slope of temperature.

All the methods so far described use natural masses to compare the earth with, and herein lies a fatal defect as regards exactness. We do not know accurately the density of these masses and what is the condition of the surrounding and underlying strata. We can really only form at the best rough guesses. Indeed, the experiments might rather be turned the other way about, and assuming the value of the mean density of the earth, we might measure the mean density of the mountain or strata of which the attraction is measured.

The Cavendish Experiment.

We turn now to a different class of experiment, in which the attracting body is altogether on a smaller scale, so that it can be handled in the laboratory. The smallness of the attraction is compensated for by the accuracy with which we know the size and mass of the attracting body.

The idea of such an experiment is due to the Rev. John Michell, who completed an apparatus for the purpose but did not live to experiment with it.

Michell's plan consisted in suspending in a narrow wooden case a horizontal rod 6 feet long, with a 2-inch sphere of lead hung at each end by a short wire. The suspending wire for the rod was 40 inches long. Outside the case were two lead spheres 8 inches in diameter. These were to be brought up opposite the suspended spheres, one on one side, the other on the other, so that their attractions on those spheres should conspire to turn the rod the same way round. Now moving each large sphere on to the other side of the case so as to pull the suspended sphere with equal force in the opposite direction, the rod should turn through twice the angle which it would describe if the spheres were taken altogether away. Hence half this angle would give the twist due to the attractions in one position alone. Knowing the torsion couple of the suspending wire for a given angle of twist and the length of the rod, the attracting force would be calculable. To find the torsion couple, Michell proposed to set the rod vibrating. From its moment of inertia and time of vibration the couple could be found.

Neglecting all corrections, the mathematics of the method may be reduced to the following:

Let the two suspended balls have mass m each, the two attracting balls mass M each. Let the rod have length $2a$ and with the suspended balls moment of inertia I; let d be the distance apart of the centres of attracting and attracted balls, and let θ be the angle through which the attraction twists the rod.

If μ is the torsion couple per radian twist, and G the gravitation constant, then

$$\mu\theta = \frac{2\mathrm{GM}ma}{d^2}.$$

The time of vibration

$$\mathrm{N} = 2\pi\sqrt{\mathrm{I}/\mu}.$$

whence, eliminating μ,

$$\frac{4\pi^2\mathrm{I}\theta}{\mathrm{N}^2} = \frac{2\mathrm{GM}ma}{d^2}.$$

Now we may obtain another equation containing G by expressing the acceleration of gravity in terms of the dimensions and density of the earth,

$$g = \mathrm{G}\,.\,\frac{4}{3}\pi\frac{r^3\Delta}{r^2} = \frac{2}{3}\mathrm{G}\Delta\mathrm{C},$$

where r is the radius, C the circumference, and Δ the density of the earth. Eliminating G between the last two equations and putting for g/π^2 the length of the seconds pendulum L—a useful abbreviation—we find

$$\Delta = \frac{3}{4}\times\frac{\mathrm{L}}{\mathrm{C}}\times\frac{\mathrm{M}ma}{d^2}\times\frac{\mathrm{N}^2}{\mathrm{I}\theta},$$

where all the terms on the right hand are known or may be measured.

On Michell's death the apparatus which he had collected for his experiment came into the possession of Prof. Wollaston, who gave it to Cavendish. Cavendish determined to carry out the experiment, with certain modifications; but he found it advisable to make the greater part of the apparatus afresh, though closely following Michell's plan and dimensions.

The actual work was done in the summer of 1797 and the following spring of 1798*

He selected for the experiment, according to Baily, an outhouse in his garden at Clapham Common, and within this he appears to have constructed an inner chamber to contain the apparatus, for he states that he "resolved to place the apparatus in a room which should remain constantly shut, and to observe the motion of the arm from without by means of a telescope," in order that inequalities of temperature and consequent air currents within the case should be avoided.

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* Experiments to determine the density of the earth. *Phil. Trans.*, lxxxviii, 1798.

The torsion rod *h h* (Fig. 11, reduced from the figure in Cavendish's paper) was of deal, 6 feet long, strengthened by a silver wire tying the ends to an upright *m g* in the middle. The two attracted balls *x x* were lead, 2 inches in diameter, and hung by short wires from the ends of the rod.

The torsion wire was $39\frac{1}{4}$ inches long, of silvered copper, and at first of such cross section as to give a time of oscillation about 15m. This was soon changed for one with a time of oscillation about 7m.

The position of the rod was determined by a fixed scale on ivory divided to $\frac{1}{20}$th inch near the end of the arm, the arm itself carrying a vernier of five divisions. This was lighted by a lamp outside the room, and was viewed through a telescope passing through a hole in the wall.

The torsion case was supported on four levelling screws. The attracting masses, lead spheres 12 inches in diameter, WW, hung down from a cross bar, being suspended by vertical copper rods. This bar could be rotated by ropes passing outside the room round a pin fixed to the ceiling in the continuation of the torsion axis.

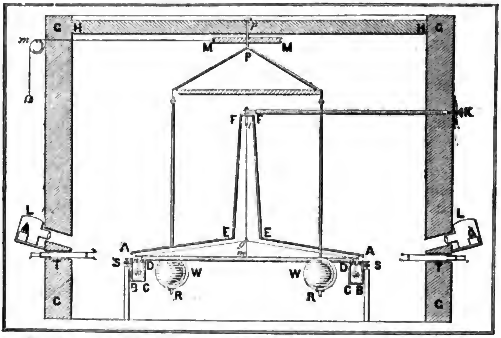

Fig. 11.—Cavendish's Apparatus. *h h*, torsion rod hung by wire *l g*; *x x*, attracted balls hung from its ends; W W, attracting masses movable round axis P. T T, telescopes to view position of torsion rod.

The masses were stopped when $\frac{1}{8}$ inch from the case by pieces of wood fastened to the wall of the building. When the masses were against the stops their centres were 8·85 inches from the central line of the case.

The method of experiment was somewhat as follows: The torsion rod was never at rest, and the centre of swing was taken as the position in which it would be if all disturbances could be eliminated. This centre of swing was determined from three succeeding extremities of vibration when the attracting masses WW were against the stops on one side. They were then swung round so as to come against the stops on the other side of the attracted masses, and the new centre of swing was observed. In a particular experiment the difference between the two centres was about six scale divisions. The time of vibration was observed from several successive passages past the centre of swing, the value obtained in the same experiment being about 427 secs., and the masses were then moved back to their first position, giving a second value for the deflection.

In computing the results various corrections had to be introduced into the equivalents of the simple formulæ which have been given above. Taking the attraction formula,

$$\mu\theta = G\frac{2Mma}{d^2},$$

a correction had to be made, because the attracting masses were not quite opposite those attracted, as the suspending bar was a little too short. Then allowance was necessary for the attraction on the torsion rod, and a negative correction had to be applied for the attraction on the more distant ball. The copper suspending rods were also allowed for, and a further correction was made for the change in attraction with change of scale reading—*i.e.*, for change of distance between attracting and attracted masses. This correction was proportional to the deviation from the central position, and may be regarded as an alteration of μ.

As to the case, it would evidently have no effect when the rod was central, but it was necessary to examine its attraction when the rod was deflected. Cavendish found that in no case did it exceed 1/1170 of the attraction of the masses, and therefore neglected it.

Turning now to the vibration formula,

$$N = 2\pi\sqrt{I/\mu};$$

this was correct when the masses were in the "midway" position—*i.e.*, in the line perpendicular to the torsion rod. But when they were in the positive or negative position, the variation in their attraction, as the balls approached or receded from them, made an appreciable alteration in the value of the restoring couple, and thus virtually altered μ. The time had therefore to be reduced by $\delta/185$ of its observed value where δ was the deflection in scale divisions due to the change of the masses from midway to near position.

But it is to be observed that, if the weights were moved from one near position to the other, and the time of vibration was taken in either position, then the same correction having to be applied to μ in both formulæ, it might be omitted from both.

In all, Cavendish obtained twenty-nine results with a mean value of

$$D = 5{\cdot}448 \pm {\cdot}033.$$

By a mistake in his addition of the results, pointed out by Baily, he gave as the mean 5·48.

Repetitions by Reich, Baily and Cornu and Baille.—His experiment has since been repeated several times. Reich made two experiments in Germany by Cavendish's method, obtaining in 1837 a value 5·49, and about 1849 a value 5·58. In England it was repeated by Baily about 1841 and 1842. Baily's experiment excited great attention at the time, and the result obtained, 5·674, was long supposed to be very near indeed to the truth. But certain discrepancies in the work gradually impaired confidence in the final result, and in 1870 MM. Cornu and Baille, in France, undertook a repetition, with various improvements and refinements. In planning out their own work they succeeded in detecting probably the chief source of error in Baily's work. They have as yet only given an interim result of about 5·5, and have shown that Baily's work, if properly interpreted, should bring out a not very different result. Their final conclusion is still to be published.

Boys's Cavendish Experiment.—In the *Philosophical Transactions* for 1895 (vol. 186, A. p. 1) is an account of a determination of the gravitation constant carried out with the greatest care by Prof. Boys. He had discovered a method of drawing exceedingly fine quartz fibres and had found them exceedingly strong and true in their elastic properties. They are therefore pre-eminently applicable in torsion experiments where small forces are to be measured. Using a quartz fibre as the torsion wire in a Cavendish apparatus, he was able to reduce the attracted weight and the whole apparatus and yet reduce the diameter of the suspending fibre so far that the sensitiveness was as great as in earlier experiments. At the same time the smallness of the apparatus allowed it to be kept at a much more uniform temperature, and the disturbances due to convection air currents were much lessened. These disturbances had much troubled the earlier workers. In Fig. 12 is a diagrammatic representation of the apparatus. The attracted masses *mm* were of gold, one pair 0·2 inch, another pair 0·25 inch in diameter. The torsion rod N was 0·9 inch long and was itself a mirror in which the reflection of a scale distant about 23 feet, and divided to 50ths of an inch, was viewed. The quartz fibre was 17 inches long.

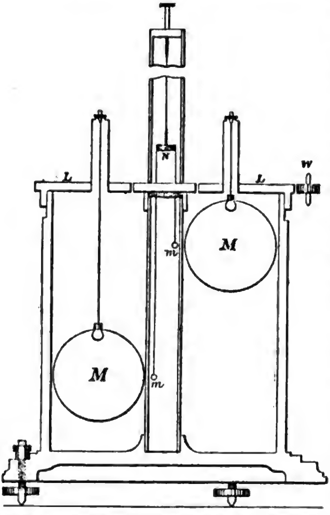

Fig. 12.—Diagrammatic Representation of a Section of Boys's Apparatus.

The attracting masses MM were lead balls $4\frac{1}{2}$ inches in diameter. Had the masses all been on one level, as in the original arrangement, with such a short torsion rod the attracting masses would have attracted both gold balls nearly equally. To avoid this, Boys had one attracting and one attracted mass at one level and the other two at a level six inches below. The balls *mm* were hung from the torsion rod by quartz fibres inside a tube about $1\frac{1}{2}$ inches diameter. The attracting masses MM were hung from the revolving lid of a concentric tubular case about 10 inches in diameter. These masses were arranged in the position in which they exerted the maximum couple on the gold balls first in one direction and then in the opposite. The deflection varied from 351 to 577 divisions, according to the balls used and the times of vibration from 188 to 242 seconds. The apparatus was most exactly constructed and measured, and the results were very concordant.

The final value, probably the best yet obtained, was:

$$G = 6{\cdot}6576 \times 10^{-8}\text{; whence } \Delta = 5{\cdot}5270$$

Braun's Experiment (*Denkschrift. der Math. Nat. Classe der Kais. Akad. Wien.* 1896. Bd. lxiv.).—In 1896 Dr. Braun published an account of an experiment carried out by him. He used the torsion-rod method, and though his apparatus was considerably larger than that of Boys, it was still much smaller than that of Cavendish, Reich or Baily. The rod was about 24 cm. long and was suspended from a tripod by a brass torsion wire nearly one metre long and 0·055 mm. in diameter. The whole torsion arrangement was under a glass receiver, about a metre high and 30 cm. in diameter, resting on a flat glass plate. The receiver could be exhausted and in the later experiments the pressure was about 4 mm. of mercury and the disturbances due to air currents were very greatly reduced. The attracted masses at the end of the rod were gilded brass spheres each weighing about 54 gms. Round the upper part of the receiver, and outside it, was a graduated metal ring which could be revolved about the axes of the torsion wire; from this were suspended, about 42 cm. apart, the two attracting masses. Two pairs were used, one a pair of brass spheres about five kgms. each, the other a pair of iron spheres filled with mercury and weighing about nine kgms. each.

Special arrangements had to be used to determine the position of the rod by means of a mirror fixed on its centre, the beam being reflected down through the bottom of the plate. The time of vibration was about 1275 secs. The result obtained was very near to that of Boys, viz.:

$$G = 6{\cdot}65786 \times 10^{-8}; \qquad \Delta = 5{\cdot}52725$$

A result very nearly the same has recently been obtained by von Eötvos (*Wied. Ann.* 59, 1896, p. 354), but he has not yet completed the work.

Wilsing's Experiment.—About 1886, Dr. Wilsing, of Potsdam, devised a modified form of Cavendish's experiment, in which a sort of double pendulum is used—*i.e.*, one with a ball below and another at a nearly equal distance above the suspension. The pendulum is then in a very sensitive state, and a very small horizontal force pulls it through a large angle.

It is then just like a torsion balance, but with a vertical instead of a horizontal rod. If weights are brought up, one to pull the upper ball to one side and the other to pull the lower ball to the other side, the pendulum twists round slightly. From the observed twist and the time of swing the attraction can be measured and compared with the pull of the earth. Wilsing found that the earth had a mean density of 5·579.

Experiments with the Common Balance.

Von Jolly's Experiment.—In 1878 and in 1881 Professor von Jolly described a method which he had devised. He had a balance fixed at the top of a tower in Munich, and from the scale-pans hung wires supporting two other scale-pans at the bottom of the tower (21 metres below). Imagine that two weights are balanced against each other at the top of the tower. If one is now brought down and put in the lower scale-pan on the same side it is nearer the centre of the earth, and, therefore, heavier. Von Jolly found a gain of about 32 milligrammes in 5 kilogrammes. He now built up a large lead sphere under the lower pan, a yard in diameter, so that its attraction was added to that of the earth. The gain on transferring the weight from the upper to the lower pan now came out to about half a milligramme more, so that the attraction of the sphere was this half milligramme. The earth's attraction was about 10,000,000 times that of the sphere, and its density was calculated to be 5·69.

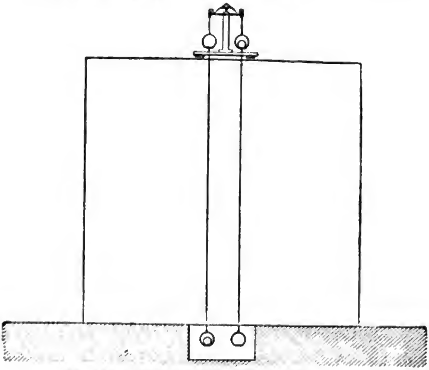

FIG. 13.—Richarz and Krigar-Menzel's Experiment.

Experiment of Richarz and Krigar-Menzel.—An experiment very much like that of Von Jolly in principle has been carried out by Drs. Richarz and Krigar-Menzel at Spandau, near Berlin (*Abhand. der Königl. Preuss Akad.* Berlin, 1898). A balance with a beam 23 cm. long was supported at a height above the floor, and from each end were suspended two pans, one near the beam the other near the floor, more than two metres lower, Fig. 13. In principle the method was as follows: Spherical gilded or platinised copper weights were used, and to begin with these were placed, say, one in the right-hand top pan, the other in the left-hand bottom pan. Suppose that in this position they exactly balanced. The weights were then moved, the right-hand one into the right lower pan, when it gained weight through the increase of gravity with a descent of over two metres; the left-hand one into the left upper pan, when it lost weight through the ascent of the same amount. The result after corrections was that the right-hand pan appeared heavier by 1·2453 mgm., half this being due to the change in position of a single kilogramme.

A lead parallelopiped was now built up of separate blocks, between the upper and lower pans, 2 metres high and 2·1 metres square, horizontally, with passages for the wires suspending the lower pans. The weighing of the kilogrammes was now repeated, but the attraction of the lead, which was reversed when a weight was moved from bottom to top, was more than enough to make up for the decrease in gravity, and the right-hand now appeared lighter on going through the same operation by 0·1211 mgm.; whence the attraction of the lead alone made a difference of 1·3664 mgm. This is four times the attraction of the lead on a single kilogramme. Knowing thus the pull of a block of lead of known form and density on the kilogramme at a known distance, and knowing too the pull of the earth on the same kilogramme, viz., 10^6 mgm., the mean density of the earth could be found.

The final result was:

$$G = 6{\cdot}685 \times 10^{-8}$$
$$\Delta = 5{\cdot}505$$

Poynting's Experiment.—The method of using the balance in this experiment will be gathered from Fig. 14. A B are two lead weights about 50 lb. each, hanging down from the ends of a very large and strong balance inside a protecting wood case. M is a large lead sphere, weighing about 350 lb., on a turn-table, so that it can move round from under A till it comes under B. The distance between the centres of M and A or M and B is about one foot. When under A, M pulls A, and so increases its weight. When moved so as to come under B the increase is taken from A and put on to B. The balance is free to move all the time, so that it tilts over to the B side an amount due to double the attraction of M on either. *m* was a balance weight half the mass of M, but at double the distance. Before this was used it was found that the movement of M tilted the floor, and the balance, which was a very sensitive level, was affected by the tilt.

To observe the deflection due to the alteration in weight, a mirror was connected with the balance pointer by the "double suspension" method, due to Lord Kelvin, and shown in Fig. 15.

With the suspension the mirror turned through an angle 150 times as great as that turned through by the balance beam. In the room above was a telescope, which viewed the reflection of a scale in the mirror, and as the mirror turned round the scale moved across the field of view. The tilt observed meant that the beam turned through rather more than 1″, and that the weight moved nearer to the mass by about $\frac{1}{8000}$ of an inch. The weight in milligrammes producing this tilt had to be found. This was done virtually (though not exactly in detail) by moving a centigramme rider about 1 inch along the beam, which was equivalent to adding to one side a weight of about $\frac{4}{10}$ milligramme. The tilt due to the transfer was observed, and was found to be very nearly the same as that due to the attraction, so that the effect of moving M round from A to B was equivalent to increasing B by $\frac{4}{10}$ milligramme, or $\frac{1}{50000000}$ of its previous weight. The pull on either is half this. In other words, the earth pulled either about 100,000,000 times as much as the mass M, and the earth, which is 20,000,000 times as far away, would at the same distance have exerted 400,000,000,000,000 times 100,000,000 times the pull, and is, therefore, so many times heavier. Thus we find that the earth weighs about $1{\cdot}25 \times 10^{25}$ lb. In obtaining the attraction of M on A or B, the attraction on the beam had to be eliminated. This was done by moving the masses A B into the positions A′ B′ one foot higher, and finding the attraction in this position. The difference was due to the change in A and B alone, for the attraction on the beam remained the same throughout.

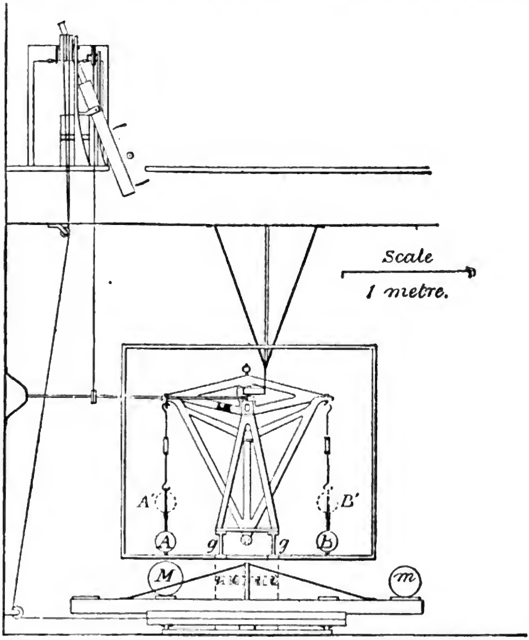

Fig. 14.—Poynting's Experiment. A B, weights, each about 50 lb., hanging from the two arms of balance. M, attracting mass on turn-table, movable so as to come under either A or B. *m*, balancing mass. A′ B′, second positions for A and B. In this position the attraction of M on the beam and suspending wires is the same as before, so that the difference of attraction on A and B in the two positions is due to the difference in distance of A and B only, and thus the attraction on the beam, &c., is eliminated.

The final result was—

$$G = 6{\cdot}6984 \times 10^{-8}$$
$$\Delta = 5{\cdot}4934$$

Experiments on the Qualities of Gravitation.

The Range of Gravitation.—The first question which arises is, whether the law of gravitation holds down to the minutest masses and distances which we can deal with. All our observations and experiments go to show that it holds throughout the long range from interplanetary distances down to the distances between the attracting bodies in the laboratory experiments described above.

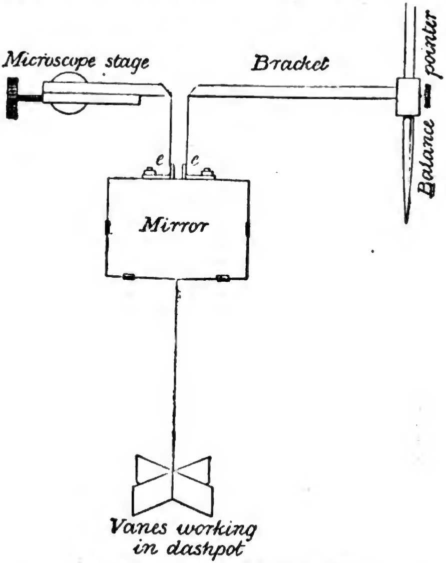

FIG. 15.—Double Suspension Mirror (half size).

The first step in the descent from celestial spaces is justified by the fact that the acceleration of gravity at the earth's surface agrees with its value on the moon, as attracted by the earth. The further step downward appears to be justified by the fair agreement of the results obtained by the various forms of Cavendish, balance, and pendulum experiments on the mean density—experiments which have been conducted at distances varying from feet down to inches. Where the law ceases to hold is yet a matter for experiment to determine. When bodies come into what we term "contact," the adhesion may possibly still be due to gravitation, according to the inverse square law, though the varying nature of the adhesion in different cases seems to point to a change in the law at such minute distances.

Gravitation not Selective.—It might be possible that some matter is attracted more than in proportion to its mass and some less. The agreement of astronomical observations with deductions from the general law is not perfectly decisive as to this possibility, for there might be such a mixture of different kinds of matter in all the planets that the general average attraction was in accordance with the law though not the attraction on each individual kind. A supposition somewhat of this description is required in an explanation which has been given of the formation of comets' tails, some matter in the comet being supposed to be acted on by the sun, not by the ordinary law but by a repulsion. This explanation is, however, now generally abandoned, an electrical origin of the tails being regarded as more probable.

But, with regard to ordinary terrestrial matter, Newton's hollow pendulum experiments (*Principia*, Book III., Prop. 6) repeated with more detail and precision by Bessel (*Versuche über die Kraft, mit welcher die Erde Körper von verschiedener Beschaffenheit anzeiht*, Abhand. der Berl. Ak. 1830, p. 41; or *Mémoires relatifs à la Physique*, tome v. pp. 71–133) prove that the earth as a whole is not selective. Still, the results might just conceivably be due to an average of equal excesses and defects. But again we may quote the various mean density experiments, and especially those made by Baily, in which a number of different attracting and attracted substances have been used with nearly the same results.

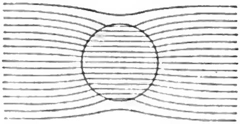

FIG. 16.—Paramagnetic Sphere placed in a previously Straight Field.

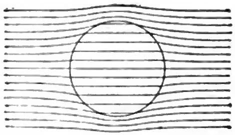

FIG. 17.—Diamagnetic Sphere placed in a previously Straight Field.

Gravitation not Affected by the Medium.—When we compare gravitation with other known forces (and those which have been most closely studied are electric and magnetic forces) we are at once led to inquire whether the lines of gravitative force are always straight lines radiating from or to the mass round which they centre, or whether, like electric and magnetic lines of force, they have a preference for some media and a distaste for others. We know, for example, that if a magnetic sphere of iron, cobalt or manganese is placed in a previously straight field, its permeability is greater than the air it replaces, and the lines of force crowd into it, as in Fig. 16. The magnetic action is then stronger in the presence of the sphere near the ends of a diameter parallel to the original course of the lines of force, and the lines are deflected. If the sphere be diamagnetic, of water, copper, or bismuth, the permeability being less than that of air, there is an opposite effect, as in Fig. 17, and the field is weakened at the ends of a diameter parallel to the lines of force, and again the lines are deflected. Similarly, a dielectric body placed in an electric field gathers in the lines of force, and makes the field where the lines enter and leave stronger than it was before.

If we enclose a magnet in a hollow box of soft iron placed in a magnetic field, the lines of force are gathered into the iron and largely cleared away from the inside cavity, so that the magnet is screened from external action.

Astronomical observations are not conclusive against any such effect of the medium on gravitation, for the medium intervening between the sun and planets approaches a vacuum,where so far we have no evidence for variation in quality, even for electric and magnetic induction. In the case of the earth, too, its spherical form might render observation inconclusive, for just as a sphere composed of concentric dielectric shells, each with its surface uniformly electrified, would have the same external field in air, whatever the dielectric constant, if the quantity of electrification within were the same, so the earth might have the same field in air whatever the varying quality of the underlying strata as regards the transmission of the action across them, if they were only suitably arranged.

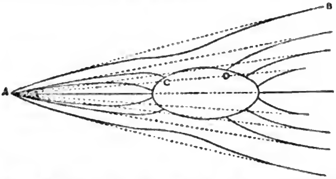

FIG. 18.—Effect of interposition of more permeable Medium in radiating Field of Force.

But common experience might lead us at once to say that there is no very considerable effect of the kind with gravitation. The evidence of ordinary weighings may, perhaps, be rejected, inasmuch as both sides will be equally affected as the balance is commonly used. But a spring balance should show if there is any large effect when used in different positions above different media, or in different enclosures. And the ordinary balance is used in certain experiments in which one weight is suspended beneath the balance case, and surrounded, perhaps, by a metal case, or, perhaps, by a water-bath. Yet no appreciable variation of weight on that account has yet been noted. Nor does the direction of the vertical change rapidly from place to place, as it would with varying permeability of the ground below. But perhaps the agreement of pendulum results, whatever the block on which the pendulum is placed, and whatever the case in which it is contained, gives the best evidence that there is no great gathering in, or opening out of the lines of the earth's force by different media.

Still, a direct experiment on the attraction between two masses with different media interposed was well worthy of trial, and such an experiment has been carried out by Messrs. Austin and Thwing.* The effect to be looked for will be understood from Fig. 18. If a medium more permeable to gravitation is interposed between two bodies, the lines of force will move into it from each side, and the gravitative pull on a body, near the interposed medium on the side away from the attracting body, will be increased.

The apparatus they used was a modified kind of Boys's apparatus (Fig. 19). Two small gold masses in the form of short vertical wires, each ·4 gm. in weight, were arranged at different levels at the ends virtually of a torsion rod 8 mm. long. They are represented in the figure by the two thickenings on the suspending fibre. The attracting masses M_1M_2 were lead, each about 1 kgm. These were first in the positions shown by black lines in the figure, and were then moved into the positions shown by dotted lines. The attraction was measured first when merely the air and the case of the instrument intervened, and then when various slabs, each 3 cm. thick, 10 cm. wide and 29 cm. high, were interposed. With screens of lead, zinc, mercury, water, alcohol or glycerine, the change in attraction was at the most about 1 in 500, and this did not exceed the errors of experiment. That is, they found no evidence of a change in pull with change of medium. If such change exists, it is not of the order of the change of electric pull with change of medium, but something far smaller. It still remains just possible, however, that there are variations of gravitational permeability comparable with the variations of magnetic permeability in media such as water and alcohol.

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* *Physical Review*, v. 1897, p. 294.

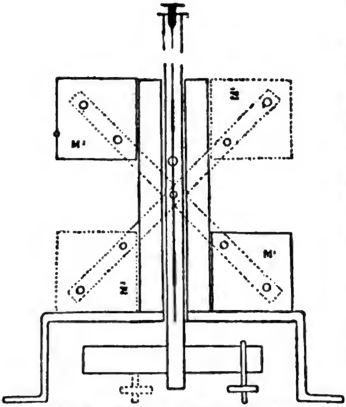

Fig. 19.—Experiment on Gravitative Permeability (Austin and Thwing).

Gravitation not Directive. — Yet another kind of effect might be suspected. In most crystalline substances the physical properties are different along different directions in a crystal. They expand differently, they conduct heat differently, and they transmit light at different speeds in different directions. We might then imagine that the lines of gravitative force spread out from, say, a crystal sphere unequally in different directions. Some years ago Dr. Mackenzie* made an experiment in America, in which he sought for direct evidence of such unequal distribution of the lines of force. He used a form of apparatus like that of Professor Boys (Fig. 12), the attracting masses being calc spar spheres about 2 inches in diameter. The attracted masses in one experiment were small lead spheres about ½ gm. each, and he measured the attraction between the crystals and the lead when the axes of the crystals were set in various positions. But the variation in the attraction was merely of the order of error of experiment. In another experiment the attracted masses were small calc spar crystal cylinders weighing a little more than ½ gm. each. But again there was no evidence of variation in the attraction with variation of axial direction.

Practically the same problem was attacked in a different way by Poynting and Gray.† They tried to find whether a quartz crystal sphere had any directive action on another quartz crystal sphere close to it, whether they tended to set with their axes parallel or crossed.

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* *Physical Review*, ii. 1895, p. 321.

† *Phil. Trans.*, 192, 1899, A. p. 245.

It may easily be seen that this is the same problem by considering what must happen if there is any difference in the attraction between two such spheres when their axes are parallel and when they are crossed. Suppose, for example, that the attraction is always greater when their axes are parallel, and this seems a reasonable supposition, inasmuch as in straightforward crystallisation successive parts of the crystal are added to the existing crystal, all with their axes parallel. Begin, then, with two quartz crystal spheres near each other with their axes in the same plane, but perpendicular to each other. Remove one to a very great distance, doing work against their mutual attractions. Then, when it is quite out of range of appreciable action, turn it round till its axis is parallel to that of the fixed crystal. This absorbs no work if done slowly. Then let it return. The force on the return journey at every point is greater than the force on the outgoing journey, and more work will be got out than was put in. When the sphere is in its first position, turn it round till the axes are again at right angles. Then work must be done on turning it through this right angle to supply the difference between the outgoing and incoming works. For if no work were done in the turning, we could go through cycle after cycle, always getting a balance of energy over, and this would appear to imply either a cooling of the crystals or a diminution in their weight, neither supposition being admissible. We are led then to say that if the attraction with parallel axes exceeds that with crossed axes, there must be a directive action resisting the turn from the crossed to the parallel positions. And conversely, a directive action implies axial variation in gravitation.

The straightforward mode of testing the existence of this directive action would consist in hanging up one sphere by a wire or thread, and turning the other round into various positions, and observing whether the hanging sphere tended to twist out of position. But the action, if it exists, is so minute, and the disturbances due to air currents are so great, that it would be extremely difficult to observe its effect directly. But the principle of forced oscillations may be used to magnify the action by turning one sphere round and round at a constant rate, so that the couple would act first in one direction and then in the other alternately, and so set the hanging sphere vibrating to and fro. The nearer the complete time of vibration of the applied couple to the natural time of vibration of the hanging sphere, the greater would be the vibration set up. This is well illustrated by moving the point of suspension of a pendulum to and fro in gradually decreasing periods, when the swing gets longer and longer till the period is that of the pendulum, and then decreases again. Or by the experiment of varying the length of a jar resounding to a given fork, when the sound suddenly swells out as the length becomes that which would naturally give the same note as the fork. Now, in looking for the couple between the crystals, there are two possible cases. The most likely is that in which the couple acts in one way while the turning sphere is moving from parallel to crossed, and in the opposite way during the next quarter turn from crossed to parallel. That is, the couple vanishes four times during the revolution, and this we may term a quadrantal couple. But it is just possible that a quartz crystal has two ends like a magnet, and that like poles tend to like directions. Then the couple will vanish only twice in a revolution, and may be termed a semicircular couple. Both were looked for, but it is enough now to consider the possibility of the quadrantal couple only.

The mode of working will be seen from Fig. 20. The hanging sphere, ·9 cm. in diameter and 1 gm. in weight, was placed in a light aluminium wire cage with a mirror on it, and suspended by a long quartz fibre in a brass case with a window in it opposite the mirror, and surrounded by a double-walled tinfoiled wood case. The position of the sphere was read in the usual way by scale and telescope. The time of swing of this little sphere was 120 seconds.

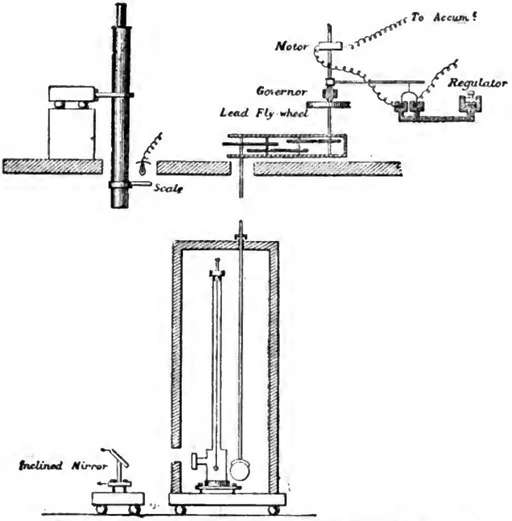

FIG. 20.—Experiment on directive Action of one Quartz Crystal on another.

A larger quartz sphere, 6·6 cm. diameter and weighing 400 gms., was fixed at the lower end of an axis which could be turned at any desired rate by a regulated motor. The centres of the spheres were on the same level and 5·9 cm. apart. On the top of the axis was a wheel with 20 equidistant marks on its rim, one passing a fixed point every 11·5 seconds.

It might be expected that the couple, if it existed, would have the greatest effect if its period exactly coincided with the 120-second period of the hanging sphere—*i.e.*, if the larger sphere revolved in 240 seconds. But in the conditions of the experiment the vibrations of the small sphere were very much damped, and the forced oscillations did not mount up as they would in a freer swing. The disturbances, which were mostly of an impulsive kind, continually set the hanging sphere into large vibration, and these might easily be taken as due to the revolving sphere. In fact, looking for the couple with exactly coincident periods would be something like trying to find if a fork set the air in a resonating jar vibrating when a brass band was playing all round it. It was necessary to make the couple period, then, a little different from the natural 120-second period, and accordingly the large sphere was revolved once in 230 seconds, when the supposed quadrantal couple would have a period of 115 seconds.

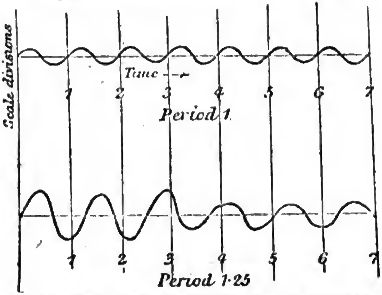

FIG. 21.—Upper Curve a regular Vibration. Lower Curve a Disturbance dying away.

Figs. 21 and 22 may help to show how this tended to eliminate the disturbances. Let the ordinates of the curves in Fig. 21 represent vibrations set out to a horizontal time scale. The upper curve is a regular vibration of range ± 3, the lower a disturbance beginning with range ± 10. The first has period 1, the second period 1·25. Now, cutting the curves into lengths equal to the period of the shorter time of vibration, and arranging the lengths one under the other, as in Fig. 22, it will be seen that the maxima and the minima of the regular vibration always fall at the same points, so that, taking 7 periods, and adding up the ordinates, we get 7 times the range, viz., ± 21. But in the disturbance the maxima and minima fall at different points, and even with 7 periods only the range is from + 16 to − 13, or less than the range due to the addition of the much smaller regular vibration.

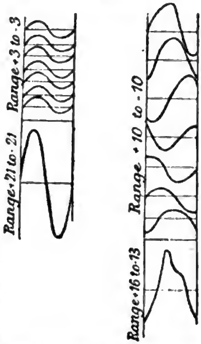

FIG. 22.—Results of Superposition of Lengths of Curves in Fig. 21 equal to the Period of the regular one.

In the experiment the couple, if it existed, would very soon establish its vibration, which would always be there, and would go through all its values in 115 seconds. An observer, watching the wheel at the top of the revolving axis, gave the time signals every 11·5 seconds, regulating the speed if necessary, and an observer at the telescope gave the scale reading at every signal, that is, 10 times during the period. The values were arranged in 10 columns, each horizontal line giving the readings of a period. The experiment was carried on for about 2½ hours at a time, covering, say, 80 periods. On adding up the columns, the maxima and minima of the couple effect would always fall in the same two columns, and so the addition would give 80 times the swing, while the maxima and minima of the natural swings due to disturbances would fall in different columns, and so, in the long run, neutralise each other. The results of different days' work might, of course, be added together.

There always was a small outstanding effect such as would be produced by a quadrantal couple, but its effect was not always in the same columns, and the net result of observations over about 350 periods was that there was no 115-second vibration of more than 1 second of arc, while the disturbances were sometimes 50 times as great. The semicircular couple required the turning sphere to revolve in 115 seconds. Here, want of symmetry in the apparatus would come in with the same effect as the couple sought, and the outstanding result was, accordingly, a little larger. But in neither case could the experiments be taken as showing a real couple. They only showed that, if it existed, it was incapable of producing an effect greater than that observed. Perhaps the best way to put the result of the work is this: Imagine the small sphere set with its axis at 45° to that of the other. Then the couple is not greater than one which would take 5¼ hours to turn it through that 45° to the parallel position, and it would oscillate about that position in not less than 21 hours.

The semicircular couple is not greater than one which would turn from crossed to parallel position in 4½ hours, and it would oscillate about that position in not less than 17 hours.

Or, if the gravitation is less in the crossed than in the parallel position, and in a constant ratio, the difference is less than 1 in 16,000 in the one case and less than 1 in 2800 in the other.

We may compare with these numbers the difference of rate of travel of yellow light through a quartz crystal along the axis and perpendicular to it. That difference is of quite another order, being about 1 in 170.

Other possible Qualities of Gravitation.—Quite indecisive experiments have been made to discover a possible alteration of mass on chemical combination.* Alterations have appeared, but they are too small and too irregular to enable any conclusion to be drawn as yet.

So far, too, there is no reason to suppose that temperature affects gravitation. Indeed, as to temperature effect, the agreement of weight methods and volume methods of measuring expansion is good, as far as it goes, in showing that weight is independent of temperature.

No research yet made has succeeded in showing that gravitation is related to anything but the masses of the attracting and the attracted bodies and their distance apart. It appears to have no relation to physical or chemical conditions of the acting masses or to the intervening medium.

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* Landolt, *Preuss. Ak. Wiss. Berlin, Sitz. Ber.*, viii. 1906, p. 266, or *Chemical News*, xciii. 1906, p. 271, has given an account of experiments which perhaps give evidence of loss of mass in the vessel containing the combining substances in certain cases. The results are very inconsistent. The loss, if proved to exist, may be due to escape through the glass, and not to alteration of mass or combination. Landolt's paper contains references to other work.

CHAPTER IV.

ELASTICITY.

CONTENTS.—Limits of Elasticity—Elastic after effect—Viscosity of Metals and Elastic Fatigue—Anomalous Effects of first Loading a Wire—Breaking Stress.

IN this chapter we shall consider changes in the conformation of solid bodies and the connection between these changes and the forces which produce them.

Many of the points with which we shall have to deal are well illustrated by the simple case of a vertical metal wire the upper end of which is fixed while the lower end carries a scale-pan. If we measure the increments of elongation of the wire when different weights are placed in the scale-pan and plot our results as a curve in which the abscissæ are the elongations of the wire—*i e.*, the extension of the wire divided by its unstretched length, and the ordinates the stretching weight (inclusive of the weight of the scale-pan) divided by the area of cross section of the unstretched wire, we obtain results similar to those shown in Fig. 23 (from *A History of the Theory of Elasticity and of the Strength of Materials*), which represents the results of experiments made by Professor Kennedy on a bar of soft steel.

The first part of the curve—when the stretching force per unit area is less than a certain value, is a straight line—*i.e.*, up to a certain point the elongation is proportional to the load per unit area of cross section,* and up to this point we find that when we remove the weight from the scale-pan the stretched wire shortens until its length is the same as it was before the weights were put on (the elongations in this stage are so small that on the scale of Fig. 23 this part of the curve is hardly distinguishable from the axis AB). When, however, we get beyond a certain point B on the curve—*i.e.*, when the stretching force per unit area is greater than the value represented by AB, the curve becomes bent, and we find on removing the weights that the wire does not return to its original length, but is permanently lengthened, and is said to have acquired *permanent set.*

The range of elongations over which the wire, when unloaded, recovers its original length, is called the range of **perfect elasticity**; when we go beyond this range we are said to exceed the **elastic limit**.

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* This seems to be only approximately true for certain kinds of iron. (*A History of the Theory of Elasticity and of the Strength of Materials.* Todhunter and Pearson, Vol. i. p. 893.

After passing the point represented by B a stage is reached where the extension becomes very large. The scale-pan runs rapidly down and the wire looks as if it were about to break. By far the greater part of this extension is permanent, and the wire, after passing the state represented by C, is not able to sustain as great a pull as before without suffering further elongation; this is shown by the bending back of the curve. The place C where this great extension begins is called the **yield-point**; it seems to be always further along the curve than the elastic limit B.

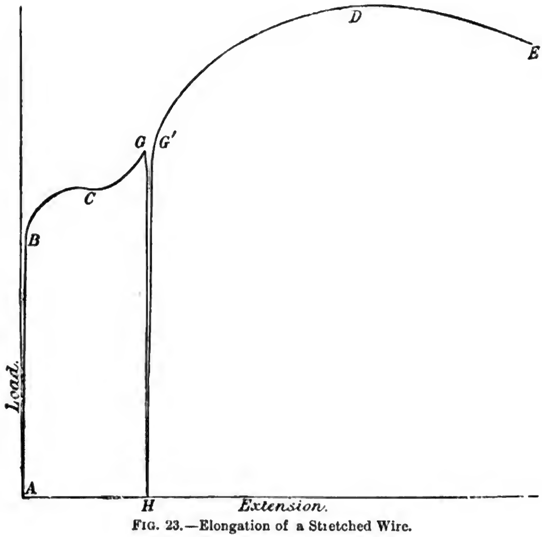


FIG. 23.—Elongation of a Stretched Wire.

The part of the increment of elongation which disappears on the removal of the stretching weight, between the elastic limit and the yield-point, is proportional to the stretching weight, and the ratio of this movement to the stretching weight per unit area is, according to the experiments of Professor Kennedy, the same as that within the limits of perfect elasticity (*see* Todhunter and Pearson's *History of Elasticity*, p. 889).

After passing the yield-point the elongation increases very rapidly with the load, and at this stage the wire is plastic, the elongation depending upon the time the stretching force acts. The extension rapidly increases and the area rapidly contracts until the **breaking-point** E is reached. The apparent maximum for the load per unit area shown in Fig. 23 is due to the contraction of the area, so that the pull per unit area of the stretched wire is no longer represented even approximately by the ordinates. About the point D the wire begins to thin down or flow locally, so that its cross section is no longer uniform, some parts being now smaller than the rest.

The portion GHG′ of the curve represents the effect of unloading and reloading at a point G past the yield-point. We see, from the shape of this portion of the curve, that the limit of perfect elasticity for this permanently stretched wire has been extended beyond the yield-point of the wire before it was permanently stretched. The range between the limit of perfect elasticity and the breaking-point is very different for different substances; for ductile substances, such as lead, it is considerable, while for brittle ones, such as glass, it is evanescent.

We are thus from our study of the loaded wire led to divide the phenomena shown by substances acted upon by forces into two divisions—one division in which the solid recovers its original form after the removal of the forces which deformed it, the other division in which a permanent change is produced by the application of the force. Even within the limits of perfect elasticity different bodies show distinct differences in their behaviour. Some recover their form immediately after the removal of the force, while others, though they recover it ultimately, take considerable time to do so. Thus a thread of quartz fibre will recover its shape immediately after the removal of the tensional and torsional forces acting upon it, while a glass fibre may, if the forces have been applied for a considerable time, be several hours before it regains its original condition. This delay in recovering the original condition of the substance is called the **elastic after-effect**; it may be conveniently studied in the case of the torsion of glass fibres.

Take a long glass fibre and fasten to it a mirror from which a spot of light is reflected on to a scale, twist the fibre about its axis and keep it twisted for a considerable time. Then remove the twisting couple: the spot of light will at once come back a considerable distance towards its old position, but will not reach it, and the rest of the journey will be a slow creep towards the old position, and several hours may elapse before the journey is completed. The larger the initial twist and the longer the time for which it was applied the greater is the temporary deflection of the spot of light from its original position.

The general shape of the curve which represents the relation between the displacement of the zero—*i.e.*, the displacement of the position of the spot of light—and the time which has elapsed since the removal of the twist, is shown in Fig. 24. In this curve the ordinates represent the displacement and the abscissæ the time since the removal of the twist.

The altitude PN, when the abscissa ON is given, depends upon the magnitude of the initial twist and the time for which it was applied; the curve is steep at first but gets flatter and flatter as the time increases. The longer the initial twist is applied the more slowly does the zero approach its original position. Very complicated movements of the zero may occur if the fibre has been twisted first in one direction and then in the opposite for a considerable number of times. The general features of this phenomenon will be illustrated by the following simple case. Suppose that immediately after the removal of the first twist, whose after-effect, if it were alone, would be represented by the curve (I), Fig. 24, a second twist in the opposite direction is applied for a time represented by ON and then removed. Suppose that the deflection of the zero due to this twist alone is represented by the dotted curve (II) (as the twist is in the opposite direction, the ordinates represent negative deflections). Then if we can superpose the effects, the displacement of the zero at a time NK after the removal of the second twist will be represented by the differences between the ordinates KR, KS of the two curves. The ordinate of the second curve may be above that of the first at the time the second twist is removed, and yet, as the curve is very steep just after the removal of the twist, curve (II) may drop down so quickly as to cut the first, as shown in the figure. Thus in this case we should have the following effects: immediately after the removal of the second twist there would be a displacement of the zero in the direction of the last applied twist, the spot of light would then creep back to the zero but would not stay there, but pass through the zero and attain a maximum deflection on the other side; it would then creep back to the zero and would not again pass through it. In this way, by superposing twists of different signs, we can get very complicated movements of the zero, which are a source of trouble in many instruments which depend upon the torsion of fibres. With quartz fibres the residual

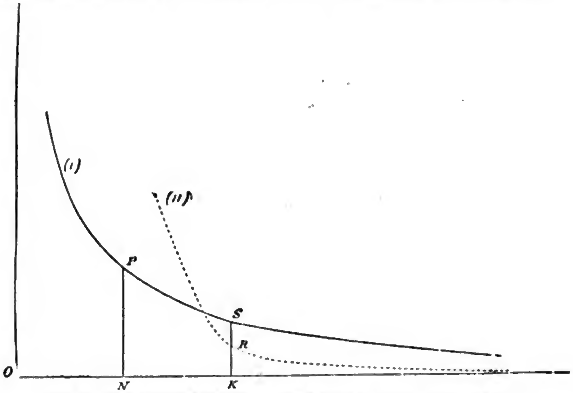


FIG. 24.—Curve showing the Elastic After-effect in a Twisted Glass Thread.

effect is exceedingly small, and this is one of the chief causes which make their use so valuable. The residual after-effect in glass is a cause of trouble in thermometry, each change of temperature causing a temporary change in the zero.

The magnitude of the elastic after-effect seems to increase very greatly when there is a want of homogeneity in the constitution of the body. In the most homogeneous bodies we know, crystals, it is exceedingly small, if it exists at all, while it is very large in glass which is of composite character, being a mixture of different silicates; it exists in metals, although not nearly to the same extent as in glass. A similar dependence upon want of uniformity seems to characterise another similar effect—the residual charge of dielectrics (*see* volume on Electricity and Magnetism), the laws of which are closely analogous to those of the elastic after-effect.

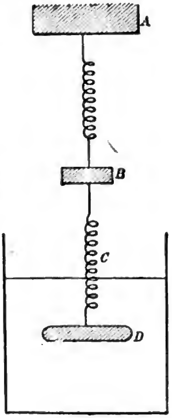


FIG. 25.

The phenomenon of elastic after-effect may be illustrated by a mechanical model similar to that shown in Fig. 25.

A is a spring, from the end, B, of which another spring C is suspended, carrying a damper D, which moves in a very viscous liquid. If B is moved to a position B′ and kept there for only a short time, so short that D has not time to move appreciably from its original position, then when B is let go it will return at once to its original zero, for D has not moved, so that the conditions are the same as they were before B was displaced. If, however, B is kept in the position B′ for a long time, D will slowly move off to a position D′, such that D′ is as much below B′ as D was below B. If now B′ is let go it will not at once return to B, for in this position the spring between B and D is extended, B will slowly move back towards its old zero, and will only reach it when the slow moving D′ has returned to D.

Viscosity of Metals and Elastic Fatigue.—If two vertical wires, one made of steel and the other of zinc, are of the same length and diameter, and carry vibration bars of the same diameter, then if these bars are set vibrating the vibrations die away, but at very different rates: the steel wire will go on vibrating for a long time, but the zinc wire will come to rest after making only a small number of vibrations. This decay in the vibrations of the wire is not wholly nor even mainly due to the resistance of the air, for this is the same for both wires; it is due to a dissipation of energy taking place when the parts of a metal wire are in relative motion, and may, from analogy with the case of liquids and gases, be said to be due to the *viscosity* of the metal. We can see that elastic after-effect would cause a decay in the vibrations of the wire. For suppose O, Fig. 26, represents the original zero—*i.e.*, the place where the force acting on the system vanishes, then if the wire is

FIG. 26.

displaced to A and then let go the new zero will be at O′, a point between A and O; thus the force will tend to stop the vibration as soon as the wire passes O′—sooner, that is, than it would do if there were no after-effect. Again, when the wire is on the other side of O, the zero will be displaced by the elastic after-effect to O″, a point between O and B, and thus again the force tending to stop the vibration will begin to act sooner than it would if there were no elastic after-effect. We can see the same thing from the study of the model in Fig. 25, for some of the kinetic energy will be converted into heat by the friction between the viscous fluid and the damper D.

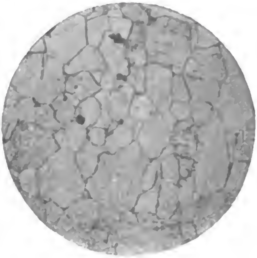


FIG. 27.

FIG. 28.

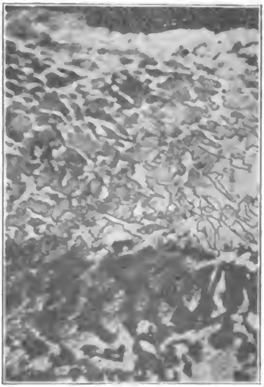


FIG. 29.

Lord Kelvin discovered a remarkable property of the viscosity of metals which he called *elastic fatigue*. He found that if a wire were kept vibrating almost continuously the rate at which the vibrations died away got greater and greater; in fact, the wire behaved as if it got tired and could only with difficulty keep on vibrating. If the wire were given a rest for a time it recovered itself, and the vibrations for a short time after the rest did not die away nearly so rapidly as they had gone just before the rest began. Muir (*Proc. Roy. Soc.*, lxiv. p. 337) found that a metal wire recovered from its fatigue if it were warmed up to a temperature above 100° C.

Anomalous Effects on first Loading a Wire.—The extension produced by a given load placed on a wire for the first time is not in general quite the same as that produced by subsequent loading; the wire requires to be loaded and unloaded several times before it gets into a steady state. The first load after a rest also gives, in general, an irregular result. It seems as if straining a wire produced a change in its structure from which it did not recover for some time.

Great light will probably be thrown on this and the other effects we have been considering by the examination by the microscope of sections of the metals. When examined in this way it is found that metals possess a structure coarse enough to be easily rendered visible. Figs. 27, 28, 29 show the appearance under the microscope of certain metals. It will be seen from these figures that in these metals we have aggregates of crystals of very great complexity—the linear dimension of these aggregates is sometimes a considerable fraction of a millimetre. These large aggregates are certainly altered by large strains. Thus Ewing and Rosenhain (*Proc. Roy. Soc.*, xlv. p. 85) have made the very interesting discovery that when a metal is strained past its yield-point there is a slipping of the crystals, which build up the aggregates along their planes of cleavage. The appearance of a piece of iron after straining past the yield-point is shown in Fig. 30; the markings in the figure are due to the steplike structure of the aggregates caused by the slipping past each other during the strain of the crystals in the aggregates, as in Fig. 31. Plasticity may thus be regarded as the yielding, or rather slipping past each other of the crystals of the large aggregates which the microscope shows exist in metals.

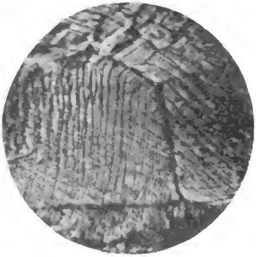
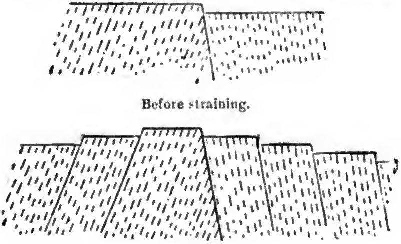


FIG. 30.



Before straining.

After straining.

FIG. 31.

In harmony with this view is the observation of McConnel and Kidd (*Proc. Roy. Soc.*, xliv. p. 331) that ice in mass is plastic when consisting of crystals irregularly arranged. In later experiments (*Proc. Roy. Soc.*, xlix. p. 323), McConnel found that a single crystal of ice is not plastic under pressure applied along the optic axis, but that it does yield under pressure inclined to the axis, as if there were slipping of the planes perpendicular to the axis.

If there is a general change in these aggregates under large strains it is possible that there are some aggregates which are unstable enough to be broken up by smaller strains, and that the first application is accompanied by a breaking up of some of the more unstable groups, so that the structure of the metal is slightly changed; we can then understand the irregularities observed when a wire is first loaded and also the existence of the elastic after-effect. Indeed, it would seem almost inevitable that any strain among such irregular-shaped bodies as those shown in Fig. 28 would result in some of them getting jammed, and thus becoming exposed to very great pressures, pressures which might be sufficient to break up some of the weaker aggregates, and thus give relief to the system. The existence of such a structure as that shown in Fig. 28 causes us to wonder whether, if a succession of very accurate observations of the elastic properties of a metal were made, the results would not differ from each other by more than could be accounted for by the errors of experiment.

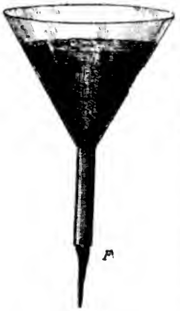


FIG. 32.

The term **viscosity** is often used in another sense besides that on p. 57. We call a substance viscous if it cannot resist the application of a small force acting for a long time. Thus we call pitch viscous because, if given a sufficiently long time, it will flow like water; and yet pitch can sustain and recover from a considerable force if this acts only for a short time. Fig. 32 shows the way in which some very hard pitch has flowed through a vertical funnel in which it has been kept in the Cavendish Laboratory for nine years. In an experiment, due to Lord Kelvin, pieces of lead placed upon a plate of pitch found in course of time their way through the plate. Many substances, however, show no trace of viscosity of this kind, for the existence of sharp impressions on old coins, the preservation of bronze statues and the like, show that metals can sustain indefinitely (or at any rate so nearly indefinitely that no appreciable change can be detected after thousands of years) their shape even under the application of small forces.

Breaking of Wires and Bars by Tension.—The following table, due to Wertheim, gives the load in kilogrammes per square millimetre necessary to break wires of different substances:

Lead	2·1	Copper . .	40·3
Tin	2·5	Platinum . .	34·1
Gold	27	Iron . . .	61
Silver . . .	29	Steel Wire . .	70
Zinc	12·8		

The process of drawing into wire seems to strengthen the material, and the finer the wire the greater is the pull, estimated per unit area of cross section, required to break it. This is shown in the following table given by Baumeister (Wiedemann, *Annalen*, xviii. p. 607):

Material.	Diameter of wire in mm.	Pull in kilogrammes per sq. mm. required to break the wire.
Swedish Iron	·72	64
" "	·50	83
" "	·30	96
" "	·25	94
" "	·15	98
" "	·10	123
Brass	·75	76
"	·25	98
"	·10	98

The effect of temperature on the pull required to break a wire is complex. Iron wire shows several maxima and minima between 15° C. and 400° C. (Pisati, *Rend. Acc. Lincei.* 1876, 76); the strength of copper, on the other hand, steadily diminishes as the temperature increases.

The strength of a material is sometimes very seriously affected by the addition of only a small quantity of another substance. Thus Sir William Roberts-Austen found that gold, to which 2 per cent. of potassium had been added, could only sustain 1/12 of the weight required to break pure gold. In the case of steel, the addition of small quantities of carbon to the iron increases the strength. The microscopical examination of the structure of metals, such as is shown in Figs. 27–30, may be expected to throw a good deal of light on effects of this kind. In this way it has been shown that the foreign substance is sometimes collected between the aggregates of the crystals of the original metals forming a weak kind of mortar, and thus greatly reducing the strength of the metal. In other cases, such as steel, a carbide is formed, and the appearance of a section of the steel under the microscope shows that the structure is much finer than in pure iron. It would seem from Sir William Roberts-Austen's experiments that the addition to gold of a metal of greater atomic volume than the gold diminishes, while a metal of smaller atomic volume increases the strength.

CHAPTER V.

STRAIN.

CONTENTS.—Homogeneous Strain—Principal Axes of Strain—Pure Strain—Elongation—Dilation or Compression—Contraction—Shear—Angle of Shear.

WHEN a body changes in shape or size it is said to be **strained**, and the deformation of the body is called **strain**.

Homogeneous Strain.—We shall restrict ourselves to the most simple class of strain to which bodies can be subjected; this is when any two lines which are equal and parallel before straining remain equal and parallel after straining. This kind of strain is called **homogeneous strain**.

Thus by a homogeneous strain a parallelogram is strained into another parallelogram, though its area and the angle between its sides may be altered by straining; parallel planes strain into parallel planes, and parallelopipeds into parallelopipeds. Figures which are similar before straining remain similar after the strain.

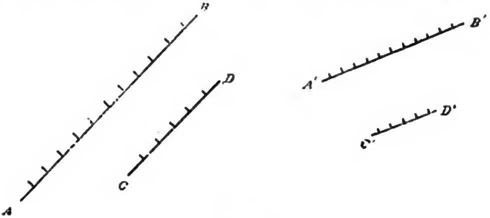


FIG. 33.

It follows from the definition of homogeneous strain that the ratio of the length of two parallel lines will be unaltered by the strain. Let AB and CD (Fig. 33) be two parallel lines. Let the ratio of AB to CD be $m : n$. Then, if m and n be commensurable, we can divide AB and CD respectively into Nm and Nn, equal parts each equal to a. Then, as before straining all these parts are equal and parallel, they will remain so after a homogeneous strain. Thus AB, after straining, will consist of Nm and CD of Nn parts, each equal to a'; and the ratio of the strained lengths is $m : n$, the same as that of the unstrained lengths. If m and n are not commensurable we can deduce the same result in the usual way by the method of limits.

From this result we can at once prove that a sphere is strained into an ellipsoid, and that three mutually perpendicular diameters of the sphere strain into three conjugate diameters of the ellipsoid. As some of our readers may not be familiar with solid geometry, we shall confine our attention to strains in one plane and prove that a circle is strained into an ellipse; the reader who is acquainted with solid geometry will not have any difficulty in extending the method to the case of the sphere. Let ABA'B' (Fig. 34) be a circle, centre C, which strains into *aba'b'*, corresponding points on the two figures being denoted by corresponding letters. Let P be a point on the circle, PL and PM parallel to CA

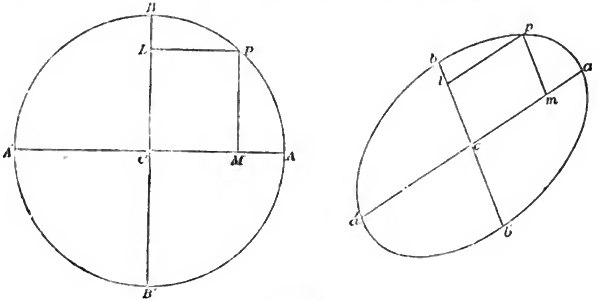


FIG. 34.

and CB respectively; let these lines on the strained figure be denoted by *pl*, *pm*.

Thus, since the ratio of parallel lines is not altered by the strain

$$\frac{\text{PL}}{\text{CA}}=\frac{pl}{ca}$$

$$\frac{\text{PM}}{\text{CB}}=\frac{pm}{cb}$$

But since P, A, B are on a circle whose centre is C

$$\frac{\text{PL}^2}{\text{CA}^2}+\frac{\text{PM}^2}{\text{CB}^2}=1$$

hence

$$\frac{pl^2}{ca^2}+\frac{pm^2}{cb^2}=1$$

or *p* is on an ellipse of which *ca* and *cb* are conjugate diameters. Thus a circle is strained into an ellipse, and two diameters at right angles to each other in the circle strain into two conjugate diameters of the ellipse. Now there are two, and only two, conjugate diameters of an ellipse (unless the ellipse degenerates into a circle) which are at right angles to each other. Hence there are two, and only two, diameters at right angles to each other before straining which remain at right angles after the strain. Now, though in general these diameters will not have the same direction after straining as they had before, yet we shall not be introducing any real limitation on the strain in so far as it affects the forces called into play by elasticity if we suppose they retain the same direction after straining as before. For, suppose OA, OB (Fig. 35), are the unstrained directions, Oa, Ob, the strained ones, we can make Oa, Ob coincide with OA, OB by rotating the strained system as a rigid body through the angle AOa. This rotation as a rigid body will not involve any relative motion of the parts of the system, and so will not call into play any forces depending upon the elasticity of the system; if, then, as is at present the case, our object is to investigate the connection between these forces and the strains, we may leave the rotation out of account.

The three directions at right angles to each other which remain at right angles to each other after straining are called the **principal axes of strain.** If these axes have the same direction after straining as before, the strain is said to be a **pure strain;** if it requires a rotation to make the principal axes after straining coincide with their position before the strain, the strain is said to consist of a pure strain and a **rotation.**

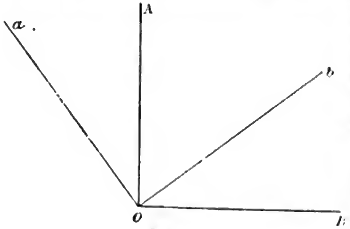


Fig. 35.

Thus the most general homogeneous strain may be resolved into extensions (regarding a compression as a negative extension) along three directions at right angles to each other. Take these directions as the axes of x, y, z respectively, then if a line of unit length parallel to the axis of x has, after the strain, a length $1+e$; one parallel to the axis of y a length $1+f$; and one parallel to the axis of z a length $1+g$, e, f, g are called the **principal elongations.** If $e=f=g$, then a sphere strains into a sphere, or any figure into a similar figure, the strained figure being an enlarged or diminished copy of the unstrained one. These cases, which are called **uniform dilatation** or **compression,** involve changes in size but not in shape.

A cube whose sides were parallel to the axes before straining and one unit in length becomes after straining a rectangular parallelopiped, whose edges are $1+e$, $1+f$, $1+g$ respectively, and whose volume is $(1+e)(1+f)(1+g)$. If, as we shall suppose all through this chapter, the elongations e, f, g are such small fractions that the products of two of them can be neglected in comparison with e, f, or g, the volume of the parallelopiped is $1+e+f+g$.

Hence the increase of unit volume due to the strain is $e+f+g$. This is called the **cubical dilatation.** We shall denote it by δ, and we have

$$\delta = e+f+g.$$

If the strain is a uniform dilatation $e=f=g$, and therefore

$$\delta = 3e$$

so that in this case the cubical expansion is three times the linear elongation.

Resolution of a Homogeneous Strain into Two Strains, one of which changes the Size but not the Shape, while the other changes the Shape but not the Size.

Let us consider the case of a strain in one plane. Let OA, OB (Fig. 36) be the principal axes of strain. Let P be the initial position of a point, P′ its position after the strain. Then if e, f are the elongations parallel to OA and OB, ξ and η the displacements of P parallel to OA and OB respectively,

$$\xi = e\mathrm{ON} = \tfrac{1}{2}(e+f)\mathrm{ON} + \tfrac{1}{2}(e-f)\mathrm{ON},$$
$$\eta = f\mathrm{OM} = \tfrac{1}{2}(e+f)\mathrm{OM} - \tfrac{1}{2}(e-f)\mathrm{OM}.$$

From these expressions we see that we may regard the strain e, f as made up of a uniform dilatation equal to $\frac{1}{2}(e+f)$, together with an **elongation** $\frac{1}{2}(e-f)$ along OA, and a **contraction** $\frac{1}{2}(e-f)$ along OB. Thus the strain superposed on the uniform dilatation consists of an expansion along one of the principal axes and an equal contraction along the other. This kind of strain does not alter the size of the body; for if σ is the elongation along OA and the contraction along OB, then a square whose sides are one unit in length and parallel to the principal axes becomes a rectangle whose sides are $1+\sigma$, and $1-\sigma$ respectively; the area of this rectangle is $1-\sigma^2$, or since we neglect the square of σ the area is unity, and thus is not altered by the strain. A strain which does not alter the size is called a **shear**. Thus any strain in one plane can be resolved into a uniform dilatation and a shear.

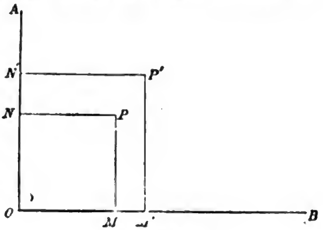


Fig. 36.

We have considered a shear as an extension in one direction and an equal compression in a direction at right angles to this; there is, however, another and more usual way of considering a shear, which may be deduced as follows:

Let OA, OB (Fig. 37) be the axes along which the extension and contraction take place. Let $\mathrm{OA} = \mathrm{OB} = \mathrm{OA}' = \mathrm{OB}' = 1$, so that before straining ABA′B′ is a square; let this square after straining be represented by $aba'b'$, which will be a parallelogram.

Since

$$\begin{aligned} \mathrm{O}a &= 1+\sigma \\ \mathrm{O}b &= 1-\sigma \\ ab^2 &= 2+2\sigma^2 \\ &= 2 \end{aligned}$$

as we suppose that σ is so small that its square may be neglected. Thus $ab = \mathrm{AB}$. Hence we can move $aba'b'$ as a rigid body and place it so that ab coincides with AB, as in Fig. 38. Then, since the area of $aba'b'$ is equal to that of ABA′B′, when the figures are placed so as to have one side in common they will lie between the same parallels. Thus, if $a''b''$ be the position of $a'b'$ when ab is made to coincide with AB, $a''b''$ (Fig. 38) will lie along A′B′; hence, except with regard to the rotation, the expansion along AO and the contraction along OB is equivalent to the strain which would bring ABA′B′ into the position AB$a''b''$. But we see that this could be done by keeping AB fixed and sliding every point in the body parallel to AB through a distance proportional to its distance from AB. We can illustrate this kind of strain by a pack of cards lying on the table, with their ends in vertical planes; now slide the cards forward, keeping the lowest one at rest in such a way that the ends are still flat although the planes are no longer vertical; each card will have been moved forwards through a distance proportional to its distance from the lowest card. The angle A′Ba'' through which a line is displaced which to begin with is perpendicular to AB is called the **angle of shear.** The plane of the shear is a plane parallel to the direction of motion and at right angles to the fixed plane.

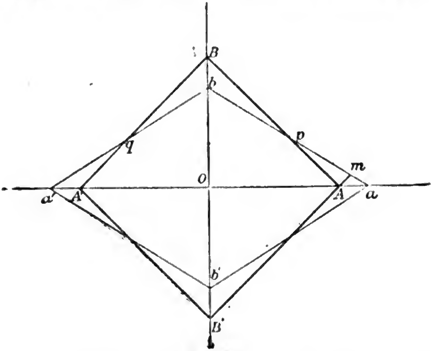


FIG. 37.

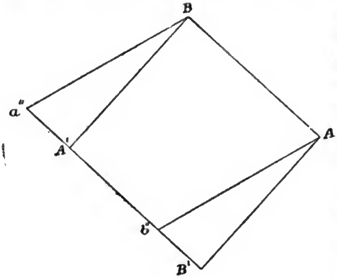


FIG. 38.

The relation between θ—the circular measure of the angle of shear—and the elongation σ along OA, and the contraction σ along OB can be found as follows. Before the rotation making ab coincide with AB, ba' makes with BA′ the angle Bqb; to make ab coincide with AB (Fig. 37) the system has to be rotated through the angle Bpb, so that after the rotation ba' will make with BA′ the angle B$q$$b$ + B$p$$b$. Now by the figure, B$q$$b$ = B$p$$b$, hence the angle of shear is 2 ∠ B$q$$b$ = 2 ∠apA. If Am is perpendicular to ap (Fig. 37), then, since the angle apA is by hypothesis small, its circular measure

$$=\frac{Am}{Ap}=\frac{Aa\sin 45}{\frac{1}{2}AO\sqrt{2}}=\frac{Aa}{AO}=\sigma,$$

hence θ, the circular measure of the angle of shear, $=2\sigma$.

If e and f are the extensions along two principal axes in the general case of homogeneous strain in two dimensions, we see from p. 65 that this strain is equivalent to a uniform dilatation $\frac{1}{2}(e+f)$ and to a shear the circular measure of whose angle is $e-f$.

CHAPTER VI.

STRESSES. RELATION BETWEEN STRESSES AND STRAINS.

CONTENTS.—General Considerations—Hooke's Law—Work required to produce any Strain—Rectangular Bar acted upon at Right Angles to its Faces.

In order that a body may be strained forces must act upon it. Consider a small cube in the middle of a strained solid, and suppose for a moment that the external forces are confined to the surface of this solid. Then the forces which strain this cube must be due to the action exerted upon it by the surrounding matter. These forces, which are due to the action of the molecules outside the cube on those inside, will only be appreciable at molecular distances from the surface of the cube, and may therefore without appreciable error be supposed to be confined to the surface. The most general force which can act on a face ABCD of the cube may be resolved into three components, one at right angles to ABCD, the other two components in the plane of ABCD, one parallel to AB, the other to BC: similarly over the other faces of the cube we may suppose similar forces to act. These forces are called ***stresses;*** the component at right angles to a face is called a ***normal stress,*** the component parallel to the face a ***tangential stress.*** The intensity of any component of the stress is the amount of the component over the face divided by the area of the face. We shall for brevity leave out the word "intensity" and speak of it simply as the ***stress.*** The dimensions of a stress are those of a force divided by an area or M/LT^2. It is measured in dynes per square centimetre; on the C.G.S. system of units the pressure of the atmosphere is about 10^6 units of stress.

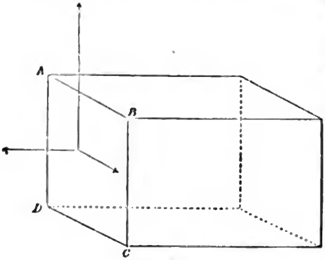


FIG. 39.

When we know the stresses over three planes meeting at a point O (Fig. 40) we can determine the stresses on any other plane through O. For let OABC be a very small tetrahedron, AOB, BOC, COA being the planes over which we know the stresses, and ABC being parallel to the plane across which we wish to determine the stress. Then as this tetrahedron is in equilibrium under the action of forces acting on its four faces, and as we know the forces over three of the faces, OAB, OBC, OCA, we can determine the force, and hence the stress, on the fourth. We need not take into account any external forces which are proportional to the volume on which they act, for the forces due to the stresses are proportional to the area of the faces, that is, to the square of the linear dimensions of the tetrahedron, while the external forces are proportional to the cube of the linear dimensions, and by making the linear dimensions of the tetrahedron exceedingly small we can make the effect of the volume forces vanish in comparison with that of the surface forces.

The stresses in a strained solid constitute a system of forces which are in equilibrium at each part of the solid with the external forces acting on the solid. If we call the external forces the load, then if a load W produces a system of stresses P, and a load W′ a system of stresses P′, then when W and W′ act together the stresses will be P + P′ if the deformation produced by either load is small.

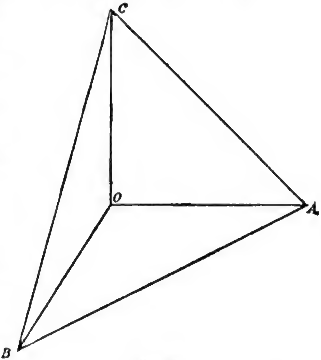


Fig. 40.

Hooke's Law.—The fundamental law on which all applications of mathematics to elasticity are based is due to Hooke, and was stated by him in the form *ut tensio sic vis*, or, in modern phraseology, that the strains are proportional to the loads. The truth of this law, when the strains do not exceed the **elastic limit** (*see* p. 53), has been verified by very careful experiments on most materials in common use. Another way of stating Hooke's Law is that if a load W produces a strain S, and a load W′ a strain S′, then a load W + W′ will produce a strain S + S′. Hence, it follows from the last article that if a system of stresses P correspond to a system of strains S, and a system of stresses P′ to a system of strains S′, then a system of stresses P + P′ will correspond to a system of strains S + S′. Hence, if we know the stress corresponding to unit strain, we can find the stress corresponding to a strain of any magnitude of the same type. Thus, as long as Hooke's law holds good, the stress and strain will be connected by a relation of the form

$$\text{Stress} = c \times \text{strain}$$

where c is a quantity which does not depend either upon the stress or the strain. It is called a **modulus of elasticity.** Thus, if the strain corresponds to a change in size but not in shape, then the stress is a uniform pressure, and the strain the diminution in volume of unit volume of the unstrained substance; in this case c is called the **modulus of elasticity of bulk**, or more frequently the **bulk modulus**. Again, if the strain is a shear which alters the shape but not the size, the strain is measured by the angle of shear and the stress by the tangential force per unit area, which must be applied to produce this shear. In this case c is called the **modulus of rigidity**. If we stretch a wire by a weight, the stress is the weight divided by the area of cross section of the wire, the strain is the increase of length in unit length of the wire, and in this case c is called **Young's modulus**. Since we can reduce the most general system of homogeneous strain to a uniform expansion or contraction and a system of shears (see p. 65) it follows that if we know the behaviour of the body (1) when its size but not its shape is changed, and (2) when its shape but not its size is changed, we can determine its behaviour under any homogeneous strain. This is true when, and only when, the properties of the substance are the same in all directions, so that a uniform hydrostatic pressure produces no change in shape, and the tangential stress required to produce a given angle of shear is independent of the plane of the shear. This statement is equivalent to saying that it only requires two moduli—*i.e.*, the bulk modulus and the modulus of rigidity, to fix the elastic behaviour of the substance, so that all other moduli, such as Young's modulus, must be expressible in terms of these two.

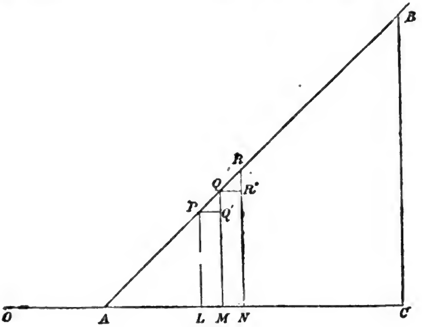


FIG. 41.

Work required to produce any Strain.—The result for the most general case, and the method by which it can be obtained, may be illustrated by considering the work required to stretch a wire. Let us suppose that the load is added so gradually that the scale-pan in which the weights are placed never acquires an appreciable velocity, so that none of the work done is converted into kinetic energy, but all is spent in stretching the wire. When this is the case, the weight in the scale-pan when in any position never exceeds by more than an infinitesimal amount the weight required to stretch the wire to that position.

Let the straight line AB, Fig. 41, represent the relation between the weight in the scale-pan and the extension of the wire, the weight being the ordinate and the extension the abscissa; let OA represent the unstretched length of the wire. Consider the work done in stretching the wire from L to M, where L and M are two points very near together. The force will be approximately equal to PL; thus the work done in stretching from L to M will be PL × LM—*i.e.*, the area PLMQ′; similarly, the work done in stretching the wire from M to N will be represented by the area QMNR′, and thus the work spent in stretching the wire from OA to OC will be represented by the sum of the little rectangular areas; but when these rectangular areas are very small, their sum is equal to the area ABC, and this equals $\frac{1}{2}$BC × AC—*i.e.*, one-half the final weight in the scale-pan × extension of the wire. Let a be the area of cross section of the wire and l the length, then $BC = a \times$ stress and $AC = l \times$ strain. Thus the work done in stretching the wire is equal to $al \times \frac{1}{2}$ strain × stress. Now al is the volume of the wire, hence the energy in each unit volume of the wire is $\frac{1}{2}$ strain × stress. Though we have considered a special case, it will be seen that the method is of general application, and that the result will hold whenever Hooke's law is true.

We have considered two ways of regarding a shear: one where the particles of the body were pushed forward by a tangential force as is represented in Fig. 38. In this case the work done on unit volume, which is the energy possessed by the sheared body, is

$$\tfrac{1}{2}T\theta,$$

where T is the tangential force per unit area and θ the angle of shear.

The other way of regarding a shear is to consider it as an extension in one direction combined with an equal contraction in a direction at right angles to the extension. Let e be the magnitude of the extension or contraction, P the pull per unit area producing the extension; this is equal to the push per unit area producing the contraction. Considering unit volume of the strained body, the work done by the pull is $\frac{1}{2}$ Pe, and that by the push is also $\frac{1}{2}$ Pe; hence the energy per unit volume is $\frac{1}{2}Pe + \frac{1}{2}Pe = Pe$, but this energy is also equal to $\frac{1}{2}T\theta$, hence

$$Pe = \tfrac{1}{2}T\theta.$$

But we know (p. 67) that $\theta = 2e$, hence

$$P = T.$$

Hence the pull or push per unit area in the one way of considering a shear is equal to the tangential stress per unit area which occurs in the other way.

If n is the coefficient of rigidity, then by the definition of n given on p. 70,

$$T = n\theta$$

hence

$$P = 2ne$$

or

$$e = \frac{P}{2n}$$

Rectangular Bar acted on by Forces at Right Angles to its Faces.—Let ABCDEFGH, Fig. 42, be a rectangular bar. Let the faces CDEF, ABGH be acted on by normal pulls equal to P per unit area, the faces ABCD, EFHG by normal pulls equal to Q per unit area, and the faces DEGB, CFHA by normal pulls equal to R per unit area. We shall proceed to find the deformation of the bar. Considering the bar as made up of rectangular parallelopipeds, with their faces parallel to the bar, we see that these will all be in equilibrium, whether they are in the interior of the bar or whether some of their faces are on the surface of the bar, if the normal stresses parallel to AC, CD, DE are respectively equal to P, Q, R, and if there are no tangential stresses. Each of these parallelopipeds will be subject to the same stresses, and will therefore be strained in the same way. Let e, f, g be the extensions parallel to P, Q, R respectively. Consider for a moment what the strains would be if the stress P acted alone: P would produce an extension proportional to P in the direction of P; let us call this λP; it would also produce contraction proportional to P in any direction at right angles to P; and if the properties of the strained substances were the same in all directions, then the contractions would be the same in all directions at right angles to P; let these contractions be μP. Then when P acts alone the *extensions* parallel to P, Q, R respectively are λP, $-\mu P$, $-\mu P$; similarly when Q acts alone the extensions in these directions are $-\mu Q$, λQ, $-\mu Q$, and when R acts alone the extensions are $-\mu R$, $-\mu R$, λR; consequently when these stresses act simultaneously we have

$$\left.\begin{aligned} e &= \lambda P - \mu Q - \mu R \\ f &= -\mu P + \lambda Q - \mu R \\ g &= -\mu P - \mu Q + \lambda R \end{aligned}\right\} \qquad (1)$$

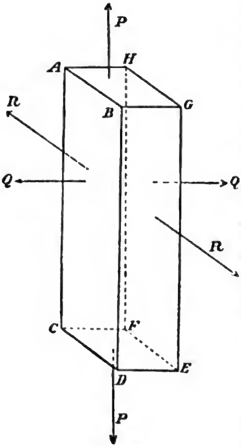


FIG. 42.

Now we have seen (p. 70) that the elastic properties of the substance are completely defined if we know the bulk modulus, which we shall denote by k, and the modulus of rigidity which we shall denote by n. Hence we must be able to express λ and μ in terms of n and k. We proceed to do this. If we apply a uniform tension to each side of the bar equal to P the dilatation of unit volume is equal to P/k, by the definition of k; but in this case the dilatation is uniform in all directions, and the linear dilatation is one-third of the volume dilatation —*i.e.*, it is equal to $P/3k$.

Hence, when $P = Q = R$, $e = f = g = \dfrac{P}{3k}$,

hence, from equations (1) $\dfrac{1}{3k} = \lambda - 2\mu$.

Let us now shear the body in the plane of PQ—*i.e.*, put $Q = -P$ and $R = 0$. In this case $e = -f = P/2n$ (see p. 71); hence by equations (1)

$$\frac{1}{2n} = \lambda + \mu.$$

Thus $\mu = \frac{1}{3}\left(\frac{1}{2n} - \frac{1}{3k}\right) = \frac{3k-2n}{18nk}$.

And $\lambda = \frac{1}{3}\left(\frac{1}{n} + \frac{1}{3k}\right) = \frac{3k+n}{9nk}$.

Young's Modulus.—A very important case is that of a bar acted on by a pull parallel to its length, while no forces act at right angles to the length. In this case $Q = R = 0$, and we have

$$e = \lambda P, f = -\mu P, g = -\mu P.$$

But in this case the stress, divided by the longitudinal strain, is called **Young's modulus**; hence, if we denote Young's modulus by q, we have,

$$P = qe, \text{ or } q = \frac{1}{\lambda} = \frac{9nk}{3k+n}.$$

This equation gives Young's modulus in terms of the bulk modulus and the rigidity.

Poisson's Ratio.—Poisson's ratio is defined to be the ratio of the lateral contraction to the longitudinal extension for a bar acted on by a stress parallel to its length. If we denote it by σ, then by this definition

$$\sigma = -\frac{f}{e}, \text{ when } Q = R = 0.$$

Thus $\sigma = \frac{\mu}{\lambda} = \frac{3k-2n}{2(3k+n)}$.

Since n is a positive quantity, we see from this expression that σ must be less than 1/2. According to a molecular theory worked out by Cauchy and Poisson, σ, for all non-crystalline substances, is equal to 1/4. The determinations of σ given in the table of elastic constants on p. 102 do not lend much support to this view.

Bar stretched longitudinally, with its Sides fixed.—The equations (1) may be written

$$e = \frac{1}{q}\Big(P - \sigma(Q+R)\Big)$$

$$f = \frac{1}{q}\Big(Q - \sigma(P+R)\Big)$$

$$g = \frac{1}{q}\Big(R - \sigma(P+Q)\Big).$$

If the bar is prevented from contracting laterally,

$$f = g = 0;$$

hence $$Q = R = \frac{\sigma P}{1-\sigma},$$

so that $$e = \frac{P}{q}\left(1 - \frac{2\sigma^2}{1-\sigma}\right).$$

Hence the elongation is less than if the sides of the bar were free in the ratio of $1 - \frac{2\sigma^2}{1-\sigma}$ to 1. In the case of a steel bar for which $\sigma = \cdot 268$ the elongation if the sides were fixed would be about 4/5 of the elongation when the sides are free.

Determination of Young's Modulus.—A simple way of measuring Young's modulus for a wire of which a considerable length is available is the following: Fix as long a length of the wire AB, Fig. 43, as is available firmly to a support. Another wire, CD, which need not be of the same material, hangs from the same support down by the side of the first wire. CD carries a millimetre scale, the length of the scale being parallel to the wire; a weight is attached to the end of this wire to keep it straight. A vernier is attached to the wire AB and moves against the scale fixed to the wire CD. The wire AB carries a scale-pan into which various weights can be placed. By reading the vernier when different weights are on the scale-pan we get the vertical depression of a fixed point on the vernier, that is of a known point on the wire, produced by a given weight. Let this depression be ϵ, when the weight in the scale-pan is increased by W. Measure the length of the wire between the fixed support and the point of attachment to the vernier; let this be l, then the elongation per unit length is ϵ/l. If ω is the cross section of wire, then the stress which produces this elongation is W/ω, so that, as Young's modulus is stress divided by strain, it is equal to

$$\frac{Wl}{\omega\epsilon}.$$

To determine the cross section, the most accurate way is to weigh a known length of the wire, first in air and then in water. The difference of the weighings in grammes will be the volume of the wire in cubic centimetres, and if we divide the volume by the length we get the cross section. Preliminary measurements should have been taken with a screw gauge to see that the wire was uniform in section. It is advisable to load and unload the wire several times before making the final measurements. This serves to straighten the wire, and avoids the anomalous results which, apart from straightening, are obtained when a wire is loaded for the first time after a rest.

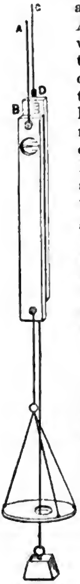


Fig. 43.

We owe the following improvements of this method to Mr. G. F. C. Searle. Two brass frames, CD, C′D′, Fig. 44, hang from the lower ends of the wires and support the two ends of a sensitive level L. One end of the level is pivoted to the frame CD by the pivots H, the other end of the level rests upon the end of a vertical screw S working in a nut attached to the frame C′D′. The two links, K, K′, prevent the frames from twisting relatively to each other about a vertical axis, but freely allow vertical relative motion. When these links are horizontal the two wires are parallel to each other. A mass M and a pan P hang from the lower ends of the frames, and the weights M and P are sufficient to straighten the wires. The connections between the wires and the frames are made by the swivels F, into which the ends of the wires are soldered. The swivels prevent the torsion of the wire. The head of the screw is divided, say, into 100 parts, while the pitch of the screw may be ·5 mm.; thus each division on the head corresponds to 1/200 mm. The measurements are made in the following way: Adjust the screw so that one end of the bubble is at zero; if a weight be placed in the pan P the wire A′ is stretched, and the bubble moves towards H; bring the bubble back to zero by turning the screw; the distance through which the screw is moved is equal to the extension of the wire.

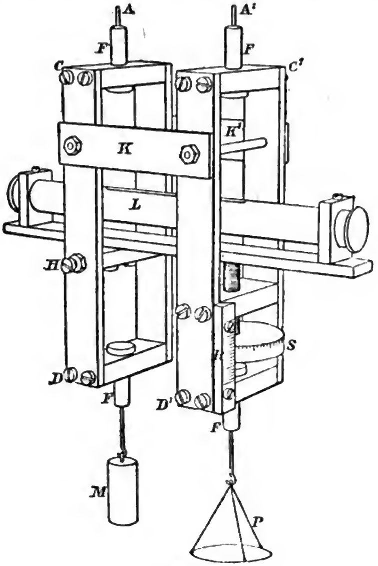


FIG. 44.

When the substance for which Young's modulus is to be determined is a bar and not a wire, the extensions obtained by any practicable weight would be too small to be measured in the way just described. In this case Ewing's extensometer may be used. This instrument is represented in Fig. 45. A is the rod whose extension is to be measured, B and C are pieces attached to A by set screws about the axes of which they revolve; the arm B′ fixed to B ends in a rounded point P, which fits into a V-shaped slot cut transversely across the end of the piece C.

Thus, when the rod A is stretched, the point P acts as a fulcrum, and Q, the opposite end of C, moves down through a distance proportional to the extension between the axes of the set screws. The displacement of Q is PQ/OP times the extension of the bar. This displacement is observed by a microscope which is attached to the bar B, and sights an object at Q. The displacement is measured by means of a micrometer scale engraved on glass in the eye-piece of the microscope; extensions of 1/20,000 of a centimetre are readily measured in this way. There is a fine screw, with a divided head between B′ and the point P. This serves to bring Q into a convenient position for sighting, and also to determine what is the absolute amount of extension corresponding to a division of the eye-piece scale; for if we know the pitch of the screw we know the displacement of Q when the screw-head is turned through one revolution; if we find how many divisions of the micrometer scale this corresponds to we can at once standardise the scale. The pull is applied to the bar by means of a small testing machine.

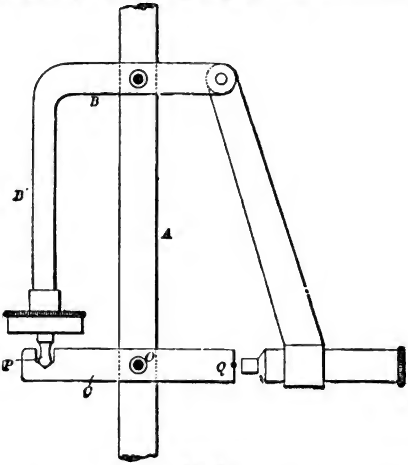


FIG. 45.

Optical Measurement of Young's Modulus.—Michelson's method of interference fringes, produced by the aid of semi-transparent mirrors, gives a very delicate way of measuring small extensions.

The principle of the method is shown in Fig. 46. A and B are plane plates of very carefully worked glass of the same thickness. One surface of A is coated with a thin film of metal, preferably platinum. The platinum may be deposited on the glass by placing the glass near a platinum cathode in an exhausted tube, and sending a current from an induction coil through the tube. The platinum sputters from the terminal and is deposited on the glass. This film is so thin as to be semi-transparent; it allows part of the light to pass through it. Suppose a beam of light, starting from S, falls on the plate A, some of it is reflected from the upper surface of the plate, and after being reflected from the mirror C returns and passes out of the plate A and enters the eye at E; another part of the beam passes through the plate A, is reflected at D, returns to the plate A, where it is reflected to E. Even when the difference of path is great, if A and B are very truly plane and of the same thickness the first part of the beam from S will interfere with the second part and produce interference bands. If the distance between one of the mirrors and the plate A is altered, the bands are shifted; an alteration of the distance through 1/4 of a wave-length will make the dark bands and light bands interchange their position; by observing the position of the bands we can measure movements of the mirror amounting to 1/50 of the wave-length of sodium light, or say a millionth of a centimetre. To apply this method to the determination of Young's modulus we keep one of the mirrors fixed while the other is carried by the wire whose extension we wish to measure. Since we can measure accurately in this way very small extensions we are able to use comparatively short wires, and so have all the conditions of the experiment under much better control than when a long wire is used. This method has been used by Mr. Shakespear at the Cavendish Laboratory. He has also used the method described on p. 43 for multiplying the small movements of the pointer of a balance, to multiply the movement due to the extension of a wire.

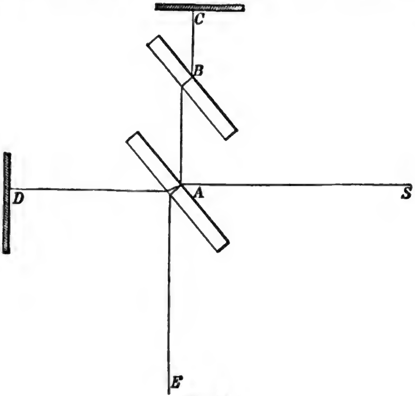


FIG. 46.

Other methods of determining q will be given in the chapter on the Bending of Rods.

CHAPTER VII.

TORSION.

CONTENTS.—Torsion of Circular Tubes and Rods—De St. Venant's Researches—Statical and Dynamical Methods of Measuring Rigidity.

Torsion of a thin Cylindrical Tube of Circular Section.—The case of a thin cylindrical tube of circular section fixed at one end and twisted by a couple whose axis is the axis of the tube, admits of a very simple solution. We can prove that each cross-section of the tube made by a plane at right angles to the axis is twisted as a rigid body in its own plane through an angle proportioned to its distance from the fixed end, and that there is no displacement of any point in the tube either radially or longitudinally. The last result follows at once from the symmetry of the tube about its axis; for from the symmetry, if the radial displacement is outwards at one part of the section it will be outwards at every point, so that there would be a swelling of the tube; reversing the couple applied to the tube would, however, reverse the displacement (since we suppose Hooke's Law to hold); hence a couple in one direction would cause the tube to swell, while one in the opposite direction would cause it to contract; it is evident, however, that whether the tube swells or contracts under a twist about its axis cannot depend upon the direction of the twist, hence we conclude that there is no radial displacement. Similar reasoning will show that the longitudinal displacement must also vanish.

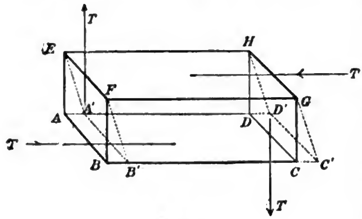


FIG. 47.

We shall now show that the tube will be in equilibrium when each cross section is twisted as a rigid body through an angle proportional to the distance of the section from the fixed end.

For suppose ABCDEFGH is a rectangular parallelopiped cut out of the tube before the twist was applied, suppose the distance between the planes ABCD, EFGH is d, and let k be the distance of the plane EFGH from the fixed end of the tube. Then, since the angle through which each section is twisted is proportional to its distance from the fixed end, if ϕ be the angle through which the section at unit distance from the fixed end is twisted, the rotation of EFGH is $k\phi$, and that of ABCD is $(k+d)\,\phi$. If a is the radius of the tube, and if t, its thickness, is small compared with a, each point in EFGH will be moved through a distance

$ak\phi$, and each point of ABCD through a distance $a\ (k+d)\ \phi$, hence after the twist the shape of the parallelopiped ABCDEFGH will be similar to EFGHA′B′C′D′, where $AA' = BB' = CC'' = DD' = ad\phi$. Hence the deformation of the elements will be a shear of which the angle of shear $= AA'/AE = a\phi$. The tangential stress T will therefore be $na\phi$. Hence the stresses on the elements will be as shown in Fig. 47, horizontal tangential stresses equal to T on the faces ABCD, EFGH, and vertical tangential stresses equal to T on the faces ABEF, CDHG. As ϕ is uniform for all parts of the tube these stresses are constant throughout the tube, and therefore each portion of the interior will be in equilibrium under these stresses. To find the condition for equilibrium under the external couple, consider a portion ABCD, Fig. 48, cut from the tube; this portion is in equilibrium under the action of the tangential stress T on its cross section, and the external couple whose moment we shall suppose is C. For equilibrium the moment of the tangential stresses round the axis must equal C. The moment of the tangential stresses is, however, T × area of cross-section of tube × radius of tube, which is equal to

$$n\phi 2\pi a^3 t;$$

hence we have $$C = n\phi 2\pi a^3 t \qquad (1)$$

which gives the rate of twist ϕ when the external couple is known.

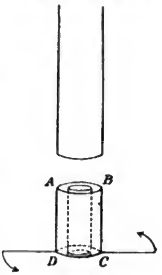


FIG. 48.

Case of a Solid Rod of Circular Section.—We can regard the rod as made up of a series of tubes, and hence from the preceding investigation we see that each cross-section of the rod will be twisted as a rigid body through an angle proportional to its distance from the fixed extremity.[*] The couple C required to twist the rod will be the sum of the couples required to twist the tubes of which it is built up, or in the notation of the integral calculus,

$$C = 2\pi n\phi \int_0^a r^3 dr$$

$$= \tfrac{1}{2}\pi n\phi a^4$$

if a is the radius of the solid cylinder. If Φ is the angle through which the lower extremity of the rod is twisted and l the length of the rod, then $l\phi = \Phi$;

hence $$C = \tfrac{1}{2}\pi n \frac{\Phi}{l} a^4$$

Thus the couple required to twist the lower end of the bar through a given angle varies directly as the fourth power of the radius and inversely as the length of the bar. If instead of a bar we have a thick tube whose inner radius is b and outer radius a, the couple C required to twist its lower extremity through an angle Φ is given by the equation

---

[*] For if the cross-sections of the different tubes were twisted through different angles, so as to shear one tube past the next, there would be twisting couples acting on the inner parts of the tube, and, since the outside of the rod is free, nothing to balance these on the outside.

$$C = \frac{1}{2}\pi n \frac{\Phi}{l}\{a^4 - b^4\}$$

The work required to twist the cylinder through an angle Φ can be shown by a method exactly similar to that given on p. 71 to be equal to $\frac{1}{2}C\Phi$; hence in the case of a solid rod the energy is

$$\frac{1}{4}\pi n \frac{\Phi^2}{l} a^4 = \frac{1}{4}\pi n l \phi^2 a^4.$$

The volume of the rod is $l\pi a^2$, hence the mean energy stored up in unit volume of the rod is $\frac{1}{4} n a^2 \phi^2$.

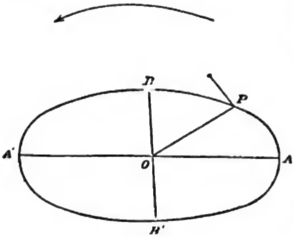


FIG. 49.

When the cross-section of the bar is not a circle the problem becomes much more difficult. It has, however, been solved by St Venant for a considerable number of sections of different shapes, including the ellipse, the equilateral triangle and the square with rounded corners. In every case except the circle a cross section made by a plane at right angles to the axis does not remain a plane after twisting but is buckled, part of the section being convex and part concave. In these cases there is a longitudinal displacement of the particles, some moving up and others down. The longitudinal movement is the same for all particles that were originally in a straight line parallel to the axis of the cylinder. We can see in the following way that there must be longitudinal displacements of the particles and find the direction of the displacement. Let us take the case when the section is an ellipse; then, if each section were rotated round the axis without any longitudinal displacement, the stress in each section at any point P would be at right angles to the line joining O to that point. Thus, if Fig. 49 represent the section of an elliptic cylinder, twisted in the direction represented by the arrow, the fixed end of the cylinder being below the plane of the paper and the twist applied to the end above the paper, the stress in the section, if there were only rotation, would be at right angles to OP; now, if P is a point on the ellipse, the tangent to the ellipse will not be at right angles to OP except at the extremities of the axes; hence in general the stress would have a component along the normal to the cylinder. Since, however, the sides of the cylinder are supposed to be free and not acted upon by forces, there cannot be equilibrium unless the stress along the normal to the cylinder vanishes; hence there must be some other displacements which will produce a stress to balance the normal component of the stress at right angles to OP. This component is directed outwards in the quadrants AB, A'B', inwards in the quadrants BA', B'A; hence the additional stress must be directed

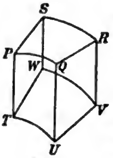


FIG. 50.

inwards in the quadrants AB, A′B′, and outwards in the quadrants BA′, B′A. Now suppose PQRSTUVW, Fig. 50, represents a parallelopiped cut from the quadrant AB, the faces PQRS, TUVW being at right angles to the axis of the cylinder and the latter nearer to the fixed end, the faces PQTU, RSVW being at right angles to OP; then there must be a stress in the plane PQRS directed from R to Q; but if there is a stress in this direction there must be a stress in RSVW parallel to RV, otherwise the parallelopiped would be set in rotation and could not be in equilibrium. Now the stress in RW parallel to RV implies either that the longitudinal displacement in the direction RV is greater than that in the same direction in the face PQTU—*i.e.*, that the longitudinal displacement increases as we recede from the axis or else that the longitudinal displacement in the opposite direction VR is less than that in the face TPQU—*i.e.*, that the longitudinal displacement diminishes as we recede from the axis. But as the longitudinal displacement vanishes at the axis itself, it seems clear that it must increase as we recede from the axis; hence we conclude that the longitudinal displacement is in the direction RV—*i.e.*, towards the fixed end of the cylinder. In the quadrant B′A′ the tangential stress at right angles to OP has a component along the outward normal, hence the longitudinal displacement is again towards the fixed end of the cylinder. In the other quadrants BA′, B′A the tangential stress has a component along the inward normal, and in this case the longitudinal displacement will be in the opposite direction—*i.e.*, *away* from the fixed end of the cylinder. Along the axis of the ellipse there is no longitudinal displacement. In Figs. 51, 52, 53, taken from De St. Venant's paper, the lines of equal longitudinal displacement are given in Fig. 51, when the cross section of the cylinder is an ellipse, in Fig. 52, when it is an equilateral triangle, and in Fig. 53, when it is a square. The dotted lines represent displacements towards the fixed end of the cylinder, the full lines displacements away from it. The direction of twist is indicated by the arrows. It will be seen that in all cases the displacement is towards the

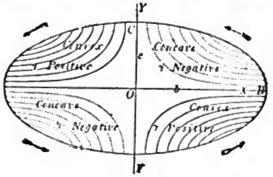


FIG. 51.

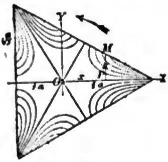


FIG. 52.

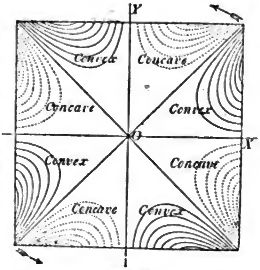


FIG. 53.

fixed end or away from it, according as the component of the tangential stress at right angles to OP along the normal to the boundary is directed to the outside or inside of the cylinder. The reason for this we saw when we considered the elliptic cylinder.

The appearance of cylinders under considerable twist is shown in Fig. 54; this case can be realised by twisting a rubber spring of elliptic or rectangular section and observing the distortion of lines drawn on the spring.

In the case of the elliptic cylinder, De St. Venant showed that the longitudinal displacement w reckoned positive when towards the fixed end of the cylinder at a point whose co-ordinates referred to the principal axes of the ellipse are x, y is given by the equation

$$w = \phi \frac{a^2 - b^2}{a^2 + b^2} xy$$

where a and b are the semi-axes of the ellipse, and ϕ the rate of twist.

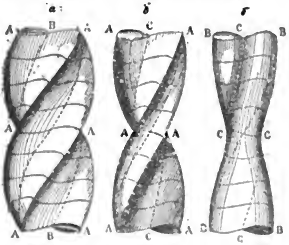


Fig. 54.

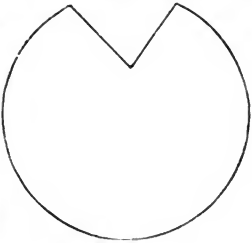


Fig. 55.

Thus the lines of equal longitudinal displacement are rectangular hyperbolas with the axes of the ellipse for asymptotes.

The couple C required to produce a rate of twist ϕ was shown by De St. Venant to be given by the equation

$$C = n\phi\pi \frac{a^3 b^3}{a^2 + b^2}$$

In the case of a thin strip of elliptic section where b is small compared with a this equation is approximately

$$C = n\phi\pi a b^3$$

Let us compare this with the couple C′ required to produce the same rate of twist in a wire of circular section, the area of the cross-section being the same as that of the strip. If r is the radius of the cross-section, then (see p. 79)

$$C' = \tfrac{1}{2} n\phi\pi r^4$$

so that

$$\frac{C}{C'} = \frac{2ab^3}{r^4}$$

Now, as the areas of the cross-sections are the same

$$\pi r^2 = \pi ab$$

hence
$$\frac{C}{C'} = \frac{2b}{a}$$

thus, as b is very small compared with a, C is small compared with C′. Thus, if we use the torsion to measure small couples, the strip will be very much more sensitive than the circular wire. Strips of thin metal are employed in some delicate torsion balances.

The greatest strain was shown by De St. Venant to be in the parts of the boundary nearest the axis—*i.e.*, the extremities of the minor axis in the case of the elliptic cylinder and the middle points of the sides in the case of the triangular cylinder.

The stress vanishes at a projecting corner, as, for example, at angles of the triangle and square. On the other hand, it becomes infinite at an internal angle, such as is shown in Fig. 55. These should, therefore, be avoided in shafts subject to torsion, or if they have to be used the angle should be rounded off.

Determination of the Rigidity by Twisting.—The coefficient of rigidity n is frequently determined by means of equation,

$$C = \tfrac{1}{2} n\pi a^4 \frac{\Phi}{l};$$

(see p. 79) which gives the relation between the couple C required to twist a circular rod of radius a and length l and the angle Φ through which the rod is twisted by the couple. The ratio of the couple to the angle may be determined (1) statically; (2) dynamically.

In the statical method a known couple is applied to the wire or rod by an arrangement such as that shown in Fig. 56, and the angle through which a pointer or mirror attached to the wire is deflected is measured. This gives C and Φ, and if we measure a and l, the preceding equation gives n.

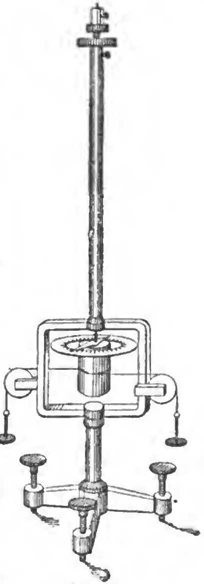


Fig. 56.

In the dynamical method for determining the rigidity, the wire whose rigidity is to be determined hangs vertically, and carries a vibration bar of known moment of inertia. If this bar is displaced from its position of equilibrium it vibrates isochronously, and the time of its vibration can be determined with great accuracy. The torsional couple tending to bring the bar back to its position of equilibrium when it is displaced through an angle Φ is equal to

$$\tfrac{1}{2}n\pi a^4\frac{\Phi}{l};$$

hence, if MK^2 is the moment of inertia of the bar, the time T of a complete vibration is given by

$$T = 2\pi\sqrt{\frac{MK^2}{\frac{1}{2}n\pi a^4/l}}$$

hence
$$n = \frac{8\pi MK^2 l}{T^2 a^4}$$

This experiment is easily made and T can be measured very accurately. The values of n found by this method are, as a rule, higher than those found by the statical method. Both methods are open to the objection that, as a occurs to the fourth power, if we make an error of 1 per cent. in the determination of a the use of the formula will lead to an error of 4 per cent. in the determination of n. Again, the use of wire in the determination of elastic constants is objectionable, as the process of wire-drawing seems to destroy the homogeneity of the metal, the outer layers differing from the inner. Unless the material is homogeneous it is not justifiable to use the equation of page 79, and any abnormality in the outer layers would seriously affect the torsion, as it is in these layers that the strain is greatest. The values of n for all metals are found to decrease as the temperature increases. (Horton, *Proc. Roy. Soc.* 73, p. 334.)

CHAPTER VIII.

BENDING OF RODS.

CONTENTS.—Bar bent into a Circular Arc—Energy in Bar—Bar Loaded at one End—Depression of End—Bar Loaded in Middle, Ends free—Bar Loaded in Middle, Ends clamped—Vibration of Loaded Bars—Elastic Curves—Stability of Loaded Pillar—Young's Modulus determined by Flexure—Table of Moduli of Elasticity.

By a rod in this chapter we mean a bar of uniform material and cross-section whose length is great compared with its transverse dimensions. We shall suppose that such a bar is acted on by two couples, equal and opposite, applied at the two ends of the rod, the plane of the couples passing through the centres of gravity of all the cross-sections of the rod, and intersecting the cross-sections in a line which is an axis of symmetry of the cross-section. Let the couples act so that the upper part of the bar is extended while the lower part is compressed. There will, therefore, be a part of the bar between the top and the bottom which is neither extended nor compressed. This part of the bar is called the *neutral surface*, and the section of it by the plane of the couple is called the *neutral axis*. Let us suppose the bar divided into thin filaments parallel to its length. We shall now proceed to show that the bar will be in equilibrium if each filament above the neutral surface is extended, each filament below that surface compressed, the extension or compression being proportional to the distance of the filament from the neutral surface, the filaments being extended or compressed as they would be if the sides of the filament were free from stress; so that if P is the tension and e the elongation, $P = qe$ where q is Young's modulus.

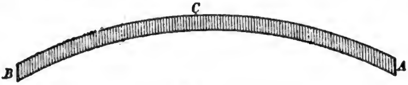


FIG. 57.

First consider the equilibrium of any filament; the strain is a uniform extension or contraction, according as the filament is above or below the neutral surface. The strain will, therefore, be a uniform longitudinal tension or compression, there will be no shearing stresses and no stresses at right angles to the length of the bar; all these statements hold whether the filament abuts on the surface or not. As the only forces acting on the filament are at right angles to its ends, and are equal and opposite, the filament will be in equilibrium. Thus each internal portion of the bar is in equilibrium, and the bar as a whole will be in equilibrium if the stresses due to the strain are in equilibrium with the external forces.

Suppose that the bar is cut at C, and that EFGH (Fig. 58) represents a cross-section of the bar, O being the centre of gravity of the section; then the forces acting on the portion CA (Fig. 57) of the bar are the external couple, whose moment we shall take to be C and the stresses acting across the cross section. Thus the condition for equilibrium is that the stresses across this section should be equivalent to a couple in the plane of bending whose moment is C. Now the tension acting on the cross-section of a filament at P is equal per unit area to qe where e is the elongation of the filament. Now e is proportional to PN if ON is perpendicular to the plane of bending and PN perpendicular to ON; let $e = a\mathrm{PN}$. Thus the force acting on the filament parallel to the length of the rod is $q.a.\mathrm{PN}\omega$ where ω is the cross-section of the filament, and the forces on all the filaments into which the bar may be supposed to be divided must be together equivalent to a couple of moment C in the plane of bending. The conditions for this are (1) that the resultant force should vanish; (2) that the moment of the forces about OM, which is perpendicular to ON, should be zero; and (3) that the moment of the forces about $\mathrm{ON} = \mathrm{C}$. All these conditions can be fulfilled if OM, ON are the principal axes of the cross-section.

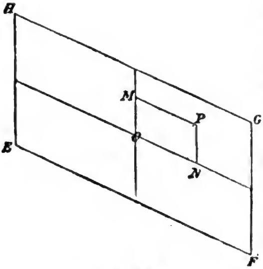


FIG. 58.

For the resultant force is $\Sigma qa.\mathrm{PN}.\omega$ where $\Sigma qa\mathrm{PN}.\omega$ denotes the sum of the product $qa.\mathrm{PN}.\omega$ for all the filaments; this vanishes since $\Sigma\mathrm{PN}\omega = 0$, O being the centre of gravity of the cross-section. The moment of these forces about OM is equal to $\Sigma qa\mathrm{PN}.\mathrm{PM}\omega$; this vanishes since $\Sigma\mathrm{PN}.\mathrm{PM} = 0$, as OM, ON are principal axes. The moment of the tension about ON is $\Sigma qa\mathrm{PN}^2\omega$; this is equal to $qa\mathrm{A}k^2$ if $\mathrm{A}k^2$ is the moment of inertia of the cross-section about ON. Hence the tensions will be in equilibrium with the external forces if $qa\mathrm{A}k^2 = \mathrm{C}$.

To find a, let us consider the deformation of a rectangle ABCD (Fig. 59) in the plane of bending, AB being a portion of the neutral axis. Let A′B′C′D′ be the strained configuration of this rectangle; then, since there is no shear, the angles at A′ and B′ will be right angles, and C′A′, D′B′ will be normals to the curve into which the neutral axis is bent; if these normals intersect in O, then O is the centre of curvature of the neutral axis. We have from the figure

$$\frac{\mathrm{C'D'}}{\mathrm{A'B'}} = \frac{\mathrm{C'O}}{\mathrm{A'O}}$$

But $\mathrm{A'B'} = \mathrm{AB}$, since the neutral axis is not altered in length by the bending, and $\mathrm{AB} = \mathrm{CD}$;

hence
$$\frac{\mathrm{C'D'} - \mathrm{CD}}{\mathrm{CD}} = \frac{\mathrm{A'C'}}{\mathrm{A'O}}$$

But if e is the elongation along CD, $e = \dfrac{\mathrm{C'D'} - \mathrm{CD}}{\mathrm{CD}}$

hence
$$e = \frac{\mathrm{A'C'}}{\mathrm{A'O}} = \frac{\mathrm{A'C'}}{\rho} = \frac{\mathrm{AC}}{\rho} \quad \text{approximately,}$$

where ρ is the radius of curvature of the neutral axis at A. But with the previous notation $e = a.\mathrm{AC}$, so that $a = \frac{1}{\rho}$.

Since $qa\mathrm{A}k^2 = \mathrm{C}$, we have $q\frac{\mathrm{A}k^2}{\rho} = \mathrm{C}$; or, $\rho = q\frac{\mathrm{A}k^2}{\mathrm{C}}$

Thus the radius of curvature of the neutral axis is constant, so that the neutral axis is a circle.

The fact that a thin bar or lath is bent into a circle by the application of two couples is often utilised for the purpose of drawing circles of large radius.

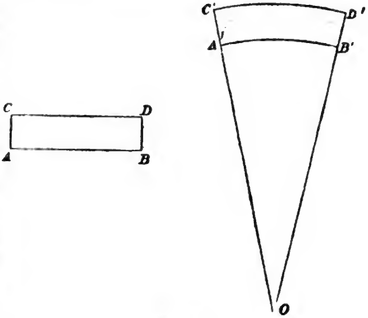


FIG. 59.

The bending of the bar will be accompanied by a change in the shape of the cross section. The elongation of the upper filaments will be accompanied by a lateral contraction equal to σ times the elongation where σ is Poisson's ratio (see p. 73), while the shortening of the lower filaments will be accompanied by a lateral expansion. Thus the shape of the cross-section supposed to be originally a rectangle will after the bending be as represented in PQLM (Fig. 60).

Suppose LM is the line where the neutral surface cuts the cross section, then the lateral contraction of PQ is equal to

$$\frac{\mathrm{LM} - \mathrm{PQ}}{\mathrm{LM}}$$

and the longitudinal extension is equal to $\frac{\mathrm{QM}}{\rho}$

hence $\frac{\mathrm{LM} - \mathrm{PQ}}{\mathrm{LM}} = \sigma\frac{\mathrm{QM}}{\rho}$

but if LP, MQ intersect in O', then $\frac{\mathrm{LM} - \mathrm{PQ}}{\mathrm{LM}} = \frac{\mathrm{QM}}{\mathrm{LO'}}$

hence $\frac{\sigma}{\rho} = \frac{1}{\mathrm{LO'}}$

But LO' is equal to the radius of curvature of the neutral surface in the plane at right angles to the length of the rod. If this is denoted by ρ' we have

$$\sigma\rho' = \rho$$

Thus the ratio of the two curvatures is equal to Poisson's ratio.

Energy in the Bar.—Consider one of the filaments into which the bar was supposed (p. 85) to be divided. Thus, if e is the elongation in this filament, l the length of the filament (which is equal to the length of the bar), ω the area of its cross-section, the energy in the filament is by p. 71,

$$\tfrac{1}{2}qe^2\omega l.$$

But $e = a.\mathrm{PN}$; hence the energy in the filament is $\tfrac{1}{2}qa^2\mathrm{PN}^2\omega l$.

The energy in the bar is the sum of the energies in the filaments, and is thus $\tfrac{1}{2}qa^2l\Sigma\mathrm{PN}^2\omega$; but $\Sigma\mathrm{PN}^2\omega = \mathrm{A}k^2$,

and $a = 1/\rho$ where ρ is the radius of curvature of the natural axis, and thus the energy is equal to $\tfrac{1}{2}q\mathrm{A}k^2l/\rho^2$.

Again, $qa\mathrm{A}k^2 = \mathrm{C}$, where C is the couple applied to the bar,

hence the energy $= \tfrac{1}{2}\mathrm{C}\dfrac{l}{\rho}$

= half the product of the couple and the angle between the tangents at the extremity of the bar. This result could be deduced at once by the method already given.

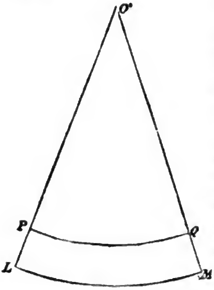


FIG. 60.

Rod bent by a Weight applied at one End.—In the case just considered the stresses in the bar were entirely normal; in this case, however, we see that for equilibrium the normal stresses must be accompanied by tangential ones. For, suppose ACB, Fig. 61, represents the bar, the weight being applied at B while A is fixed; consider a section through C made by a plane at right angles to the length of the bar. Then the portion CB of the bar must be in equilibrium under the action of the stresses across the section at C and the weight W at the end of the bar; thus the stresses across C must be equivalent to a vertically upward force

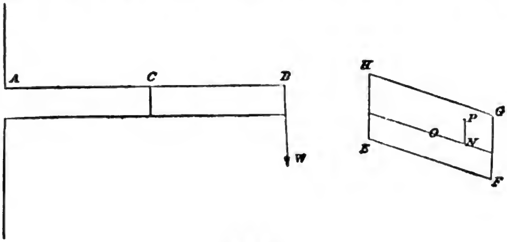


FIG. 61.

W and a couple whose moment is W.BC: there must be, therefore, tangential stresses acting across the section whose resultant is a force W acting upwards. We shall show, however, that if the lateral dimensions of the bar are very small, then, except quite close to the end B, the tangential stresses will be very small compared with the normal stresses. For let EFGH represent a section of the bar, O the centre of the section, and ON an axis at right angles to the plane of bending. Then, if A is the area of the cross-section, T the average tangential stress over the area

$$TA = W$$

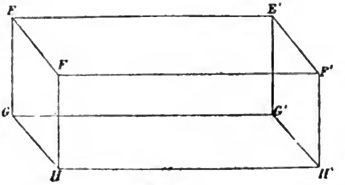


Fig. 62.

Let N represent the normal stress at a point P, $d\omega$ a small area round P, then since these normal stresses are equivalent to a couple whose moment round ON is W.BC, we have

$$\int N.PN d\omega = W.BC.$$

Thus the average normal stress must be of the order of magnitude

$$\frac{W.BC}{Ad}$$

where d is a quantity comparable with the depth of the bar. Hence, since $\frac{W}{A} = T$, the magnitude of N is comparable with $T \times BC/d$, so that if the distance of the section from the end is large compared with the lateral dimensions of the bar, the normal stresses will be very large compared with the tangential ones. In the subsequent work we shall confine our attention to the effect of the normal stresses, but this must be regarded as an approximation only applicable to very thin rods. Let Fig. 62 represent a small rectangular parallelopiped cut out of the bar, the faces EFGH, E'F'G'H' being at right angles to the length of the bar, while the faces FF'H'H, EE'GG' are parallel to the plane of bending, then the actual state of stress may be thus described. The normal stresses are confined to the faces EFGH, E'F'G'H', there being no normal stresses over the other faces; there are tangential stresses on the faces FF'HH', EE'GG', and also on the faces GG'HH' and EE'FF', but there are no tangential stresses over the faces EFGH, E'F'G'H'.

We may proceed to find the bending of the rod produced by the weight at its end in the following way. Suppose PQRS (Fig. 62A) represents a portion of a rod bent as on p. 85, by couples of moment C acting at its ends, then the stresses in the bar are such as to cause a couple with moment C to act across PQ and a couple whose moment is C to act across the section RS. The stresses which produce these couples, as we have seen on p. 87, correspond to a state of strain such that the central axis of the portion of the bar is bent into a circle whose radius ρ is given by the equation

$$q\frac{Ak^2}{\rho} = C.$$

Now suppose that PQRS, instead of being a portion of a bar acted on by a couple, is a portion of one acted on by a force at the end A: then neglecting, for the reasons given above, the tangential stresses across the section, the stresses are equivalent to a couple W.AN across the section PQ and a couple W.AM across the section RS, and as AN and AM differ but little from AL where L is the middle point of MN, we may regard the ends of PQRS as being acted on by equal and opposite couples whose moment is W.AL. Hence, by what we have just seen, the central axis of PQRS will be bent into the arc of a circle whose radius ρ is given by the equation

$$q\frac{Ak^2}{\rho} = W.AL;$$

hence, when the bar is acted on by a weight applied at one end, the neutral axis of the bar is bent into a curve such that the radius of curvature at a point varies inversely as the distance of the point from the end to which the weight is applied.

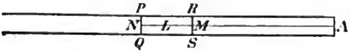


FIG. 62A.

Depression of the Bar; Angle between Tangents at two Points on the neutral Axis.—Suppose Fig. 63 represents the curved position of the neutral axis.[*] Suppose RS are two points near together on the neutral axis, then the angle between the tangents at R and S is equal to RS/ρ where ρ is the radius of curvature of RS; but $1/\rho$ is equal to $W.AR/q.Ak^2$, hence $\Delta\theta$ the angle between the tangents at R and S is equal to

$$\frac{W}{q.Ak^2}AR.RS$$

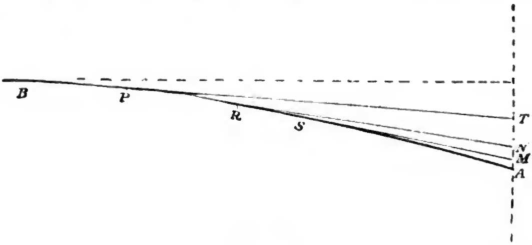


FIG. 63.

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[*] Though this figure shows for clearness' sake considerable curvature, yet it must be remembered that in all these investigations we are only dealing with cases in which the bending is very slight and the neutral axis consequently nearly straight.

or, in the notation of the differential calculus, if $s = \text{AR}$, we have

$$\Delta\delta = \frac{\text{W} \,.\, s}{q.\text{A}k^2}\, ds \tag{1}$$

hence δ, the angle between the tangents at A and P, is given by the equation

$$\delta = \int_0^{\text{AP}} \frac{\text{W}}{q.\text{A}k^2}\, s ds \tag{2}$$

$$= \tfrac{1}{2} \frac{\text{W}}{q.\text{A}k^2}\, \text{AP}^2$$

Suppose the tangent at P cuts the vertical through A in the point T, we shall proceed to find an expression for AT. Let the tangents at R,S cut the vertical line through A in the points M,N, then, remembering that these tangents are very nearly horizontal, we have approximately, if $\Delta\delta$ is the angle between the tangents at R and S,

$$\text{MN} = \text{AR} \,.\, \Delta\delta = \frac{\text{W} \,.\, s^2}{q.\text{A}k^2}\, ds \qquad \text{by (1)}$$

$$\text{Now AT} = \Sigma\text{MN} = \int_0^{\text{AP}} \frac{\text{W} \,.\, s^2}{q\text{A}k^2}\, ds = \frac{\text{W AP}^3}{3 \times q\text{A}k^2} \tag{3}$$

If the end B of the bar is clamped so that the tangent is horizontal, then the distance between A and the point where the vertical through A cuts this tangent will be the vertical depression of A produced by the weight W; hence, if d be this depression, we have by (3)

$$d = \frac{\text{W}}{3q\text{A}k^2}\, \text{AB}^3 \tag{4}$$

Thus the vertical depression of the end is proportional to the weight, to the cube of the length, and inversely proportional to the moment of inertia of the cross section about an axis through its centre at right angles to the plane of bending; it is also inversely proportional to the value of Young's modulus for the material of which the bar is made.

Since the depression is proportional to the weight, the energy stored in the bar is equal $\frac{1}{2}\text{W}d$, and this by equation (4) is equal to

$$\tfrac{1}{6} \frac{\text{W}^2}{q.\text{A}k^2}\, \text{AB}^3$$

We shall now proceed to find the depression PM (Fig. 64) of any point P on the bar below the horizontal tangent at B. Let the tangent to the central axis at P cut the vertical line through A in the point T, and let the horizontal line through P cut this line at O; then the vertical depression of P is

$$\text{PM} = \text{AN} - \text{AT} - \text{TO}$$

Now TO = PO × angle the tangent at P makes with the tangent at B, and since PO is approximately equal to AP, and the tangent at A makes with the tangents at P and B angles whose circular measures are respectively $W.AP^2/2qAk^2$ and $W.AB^2/2qAk^2$ (by equation (2)), we have

$$TO = \frac{AP.W}{2qAk^2}(AB^2 - AP^2)$$

By equation (3) we have

$$AN = \frac{W}{3qAk^2}AB^3$$

Thus

$$AT = \frac{W}{3qAk^2}AP^3$$

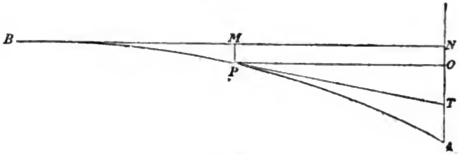

Fig. 64.

Hence

$$PM = \frac{W}{qAk^2}\left\{\frac{AB^3 - AP^3}{3} - \frac{AP(AB^2 - AP^2)}{2}\right\}$$

$$= \frac{W}{qAk^2}BP^2\left\{\frac{3AP + 2BP}{6}\right\} \qquad (5)$$

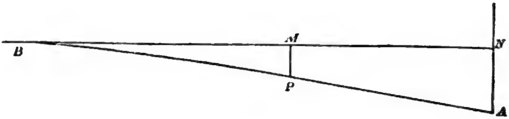

Fig. 65.

Let us now find what would be the depression of A if the weight W were applied at P. In this case AP would be straight, and if AN. Fig. 65, is the depression of A,

AN = PM + AP × angle which tangent at P makes with the horizontal

Now by (4)

$$PM = \frac{W}{3qAk^2}BP^3$$

and by (2) the angle the tangent at P makes with the horizontal is equal to

$$\frac{W}{2qAk^2}BP^2$$

hence

$$AN = \frac{W}{qAk^2} BP^2 \left\{ \frac{BP}{3} + \frac{AP}{2} \right\}$$

$$= \frac{W}{qAk^2} BP^2 \left\{ \frac{3AP + 2BP}{6} \right\} \qquad (6)$$

Comparing equations (5) and (6) we see that the depression at P when the load is applied at A is the same as the depression at A when the load is applied at P. In the case we have just been considering one of the points is at the end of the rod. The theorem, however, is a general one, and holds wherever the points A and P may be.

The relation between the depression and the weight given by equation (4) gives us a means of determining q by measuring the flexure of a beam. In experiments made with this object, however, it has been more usual to use the system considered in the next paragraph, that of a beam supported at the ends and loaded in the middle.

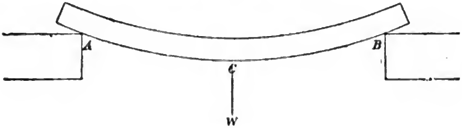

FIG. 66.

Beam Supported at the Ends and Loaded in the Middle.—The ends of the beam (Fig. 66) are supposed to rest on knife edges in the same horizontal line. The tangent at C, the middle point, is evidently horizontal, and the pressure on each of the supports is W/2. Considering now the portion AC of the rod, it has the tangent at C horizontal, and it is acted upon by a vertical force equal to W/2 at A. The conditions are the same as for a rod of length AC clamped at C and acted on by a vertical force W/2, the case just treated; hence by equation (4) d, the vertical distance between A and C, is given by the equation

$$d = \frac{W}{2qAk^2} \frac{AC^3}{3}$$

$$= \frac{W}{48qAk^2} AB^3$$

Rod Clamped at both Ends and Loaded in the Middle.—Suppose AB is a rod loaded at C, its middle point, and clamped at the ends A and B, which are supposed to be in the same horizontal line.

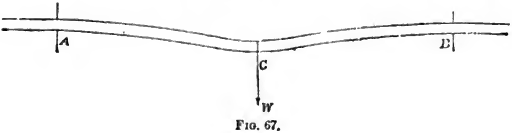

FIG. 67.

The action of the supports A, B on the rod will be equivalent to a vertical force and a couple. The magnitude of the vertical force is evidently W/2 if W is the weight at C. We can find the value of the couple Γ as follows. By the action of the force W/2 alone the tangent to the neutral axis at A would make, with the tangent at C, an angle whose circular measure is

$$\frac{W}{2qAk^2}\frac{AC^2}{2}$$

But since the tangent at A is parallel to the tangent at C, the couple must bend the bar so that if it acted alone the tangent at A would make with that at C an angle equal and opposite to that just found. Through a couple Γ applied to the bar the tangents at A and C would make with each other an angle whose circular measure is

$$\frac{\Gamma}{qAk^2}AC$$

hence
$$\frac{W}{2qAk^2}\frac{AC^2}{2}=\frac{\Gamma}{qAk^2}AC$$

or
$$\Gamma=\tfrac{1}{4}W.AC.$$

To find the depression of the middle point, we consider the effect of the force W/2, and the couple Γ separately. In consequence of the action of the force W/2, the middle point, C would by equation (4) be depressed below the line AB by

$$\frac{W}{2qAk^2}\frac{AC^3}{3}$$

The couple Γ would bend the bar into a circle whose radius ρ is qAk^2/Γ. This would raise the point C *above* A by

$$\frac{AC^2}{2\rho}$$

i.e., by
$$\frac{\Gamma AC^2}{2qAk^2}=\frac{W}{2qAk^2}\frac{AC^3}{4}$$

The depression of C when both the force and the couple act is therefore

$$\frac{W}{2qAk^2}\frac{AC^3}{3}-\frac{W}{2qAk^2}\frac{AC^3}{4}$$

$$=\frac{W}{24qAk^2}AC^3=\frac{WAB^3}{192qAk^2}$$

The depression of the middle point of the bar when the ends are fixed is thus only 1/4 of the depression of the same bar when the ends are free.

Vibration of Loaded Bars.—Since the deflection of the bar is in all cases proportional to the deflecting weight, a bar when loaded will execute isochronous vibrations, the time of a complete vibration being equal to

$$2\pi\sqrt{M/\mu},$$

where M is the mass of the load and μ the force required to produce unit depression. From the preceding investigations we see that $\mu = p.qAk^2/l^3$ where l is the length of the bar and p a numerical factor, which is equal to 3 when the weight is applied at the end of the bar, to 48 when the weight is applied at the middle point of a bar with its ends free, and to 192 when the load is applied to the middle point of a bar with its ends clamped.

To take a numerical example. Let us suppose we have a steel bar 30 cm. long, 2 cm. broad, and ·2 cm. deep, loaded at the end with a mass of 100 grammes. Then since for steel $q = 2{\cdot}139 \times 10^{12}$, and in this case $M = 100$, $p = 3$, $l = 30$, $A = {\cdot}4$, $k^2 = \frac{1}{3}\,({\cdot}1)^2 = {\cdot}0033$, we find by substituting in the formula that the time of vibration is about $\frac{1}{8}$ of a second.

To take another case, suppose a man weighing 70 kilogrammes stands on the middle of a wooden plank 4 metres long, 30 cm. wide, and 4 deep, supported at its ends, what will be the time of swing? For wood we may take $q = 10^{11}$; putting $p = 48$, $M = 7 \times 10^4$, $l = 4 \times 10^2$, $A = 120$, $k^2 = \frac{1}{3}\,(2)^2 = 1{\cdot}33$, we find that the time of swing is about ·5 seconds.

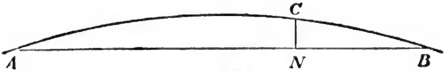

Fig. 68.

Elastic Curve.—Let us now consider a case like that of a bow where the force is parallel to the line joining the ends of the bar. Consider the equilibrium of the portion CB (Fig. 68) under the stresses at C, and the tension T in the string at B.

Thus the stresses across C must be equivalent to a couple T.CN and a force T, CN being the perpendicular from C on the line of action of the force. Confining our attention to the couple, we see that if ρ is the radius of curvatures at C of the neutral axis of the rod,

$$q\frac{Ak^2}{\rho} = \text{T.CN}$$

where q is Young's modulus for the rod, Ak^2, the moment of inertia of the cross-section of the rod about an axis through its centre at right angles to the plane of bending. From equation (7) we see that $1/\rho$ is proportional to CN; hence the curve into which the central axis is bent is such

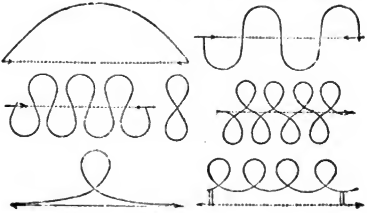

Fig. 69.

that the reciprocal of the radius of curvature at any point is proportional to the distance of the point from a straight line. Curves having this property are called *elastic curves* or *elasticas;* curves such as those shown in Fig. 69 are included in this family; they may be produced by taking a flexible metal ribbon, such as a watch-spring, and pushing the ends together. One of these curves is of especial importance—viz., the one where the distance of any point on the bent rod from the line of action of the force is very small. We shall show that this curve is the path of a point near the centre of a circle when the circle rolls on a straight line. To prove this it is only necessary to show that the reciprocal of the radius of curvature of this path is proportional to the distance from the straight line which is the path of the centre of the circle. Let us suppose that the circle rolls with uniform angular velocity ω along the straight line. Let C be the centre of the circle, P any position of the moving point, G the point of contact of the circle with the line along which it rolls, PN the perpendicular on GC. Then if v be the velocity of the point, ρ the radius of curvature of the path,

$$\frac{v^2}{\rho} = \text{acceleration of P along the normal to its path} \qquad (8)$$

Now since the circle rolls on the line without slipping the velocity of G is zero. Hence the system is turning about G, so that the velocity at P is at right angles to PG and equal to ωPG; hence PG is the normal to the path and

$$v = \omega.\text{PG}.$$

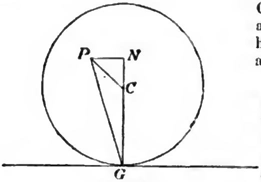

FIG. 70.

Now the acceleration of P is equal to the acceleration of C plus the acceleration of P relative to C; since C moves uniformly along a straight line the acceleration of C is zero, and since P describes a circle round C, the acceleration of P relative to C is equal to ω^2CP and is along PC. Thus the acceleration of P along the normal to its path is equal to

$$\omega^2\text{CP}\cos\text{CPG}$$

and we have therefore by (8)

$$\frac{\omega^2\text{PG}^2}{\rho} = \omega^2\text{CP}\cos\text{CPG}$$

or

$$\frac{1}{\rho} = \frac{\text{CP}\cos\text{CPG}}{\text{PG}^2}$$

Since the angle PGC is very small, the angle CPG is very nearly equal to the angle PCN, and PG is very nearly equal to a, the radius of the rolling circle; hence approximately

$$\frac{1}{\rho} = \frac{\text{CP}\cos\text{PCN}}{a^2} = \frac{\text{CN}}{a^2}$$

Thus $1/\rho$ is proportional to the distance of P from the straight line described by C.

From the equation

$$q\frac{Ak^2}{\rho} = \text{T.CN}$$

we see that

$$a^2 = \frac{qAk^2}{\text{T}}$$

The shape of the curve is shown in Fig. 71. The distance between two points of inflection, that is, between two points, such as A and B, where $1/\rho$ vanishes, is equal to πa.

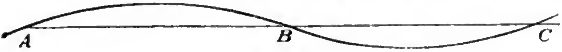

Fig. 71.

Stability of a loaded Pillar.—The preceding result at once gives us the condition that a vertical pillar with one end fixed vertically in the ground should not bend when loaded with a weight W—*i.e.*, the condition that the pillar should be stable. For, suppose the pillar bends slightly, assuming the position AB, Fig. 72, then AB is an elastica and B must be a point of inflection, while, since A is fixed vertically in the ground, the tangent at A is parallel to the line of action of the force. The distance—measured parallel to the base-lines—between a point of inflection and the point where the tangent is parallel to the base-line is half the distance between two points of inflection, and is, therefore, equal to $\frac{1}{2}\pi a$, or, substituting the value of a, to

$$\tfrac{1}{2}\pi\sqrt{\frac{qAk^2}{W}}$$

where W is the weight; hence, in order that the pillar should be able to bend, l, the length of the pillar, must not be less than

$$\tfrac{1}{2}\pi\sqrt{\frac{qAk^2}{W}}$$

or, in order to avoid bending,

$$W < \frac{\pi^2 qAk^2}{4l^2} \qquad (9)$$

If the cross-section of the pillar is a circle of radius b, then $Ak^2 = \frac{1}{4}\pi b^4$. Thus the weight which a vertical pillar can support without becoming unstable is proportional to the fourth power of the radius and inversely proportional to the square of the length of the pillar. To take a special case, let us consider a steel knitting-needle, 20 cm. long and ·1 cm. in radius and take $q = 2{\cdot}14 \times 10^{12}$. We find W less than $1{\cdot}04 \times 10^6$—*i.e.*, less than about 1056 grammes.

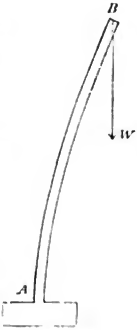

Fig 72.

If the rod, instead of being fixed at one end, is pressed between two supports so that the ends are free to bend in any direction, Fig. 73, the ends must be points of inflection, the distance between which is πa or

$$\pi\sqrt{qAk^2/W};$$

hence
$$l=\pi\sqrt{\frac{qAk^2}{W}}$$

in the limiting case when the pillar can bend. Hence for stability

$$W<\frac{\pi^2 qAk^2}{l^2} \tag{10}$$

In the case where both ends are fixed (as in Fig. 74), the tangents at

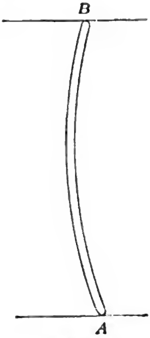

Fig. 73.

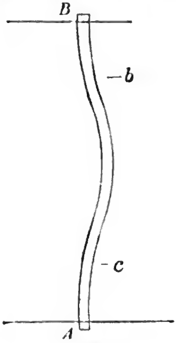

Fig. 74.

the ends must be parallel to the line of action of the force, and there must be two points of inflection at, b and c; hence the distance between the ends is twice the distance between two points of inflection, so that

$$\begin{aligned} l &= 2\pi a \\ &= 2\pi\sqrt{\frac{qAk^2}{W}} \end{aligned}$$

Hence for stability
$$W<\frac{4\pi^2 qAk^2}{l^2} \tag{11}$$

Comparing (9) and (11), we see that a rod with both ends fixed will, without buckling, support a weight sixteen times greater than if one end were free.

Since a pillar can only support without buckling a finite weight, and as this weight diminishes as the length of the pillar increases, it follows that a pole of given cross-section would, if high enough, begin to bend under its own weight, so that there is a limit to the height of a vertical pillar or tree of given cross-section. Suppose W is the weight of the pillar, and suppose as an approximation that the problem is the same as if the weight were applied at the middle point of the pillar, then if l is the length of the pillar we see from (9) that

$$W < \frac{\pi^2 q A k^2}{l^2}$$

or

$$l < \pi\sqrt{\frac{qAk^2}{W}}$$

A more accurate investigation, which requires the aid of higher mathematics, shows that the accurate relation is

$$l < 2{\cdot}8\sqrt{\frac{qAk^2}{W}}$$

Let us take the case of a pine tree of uniform circular section from top to bottom, let the diameter of the tree be 15 cm. For deal $q = 10^{11}$, and taking the specific gravity of deal as ·6, we have

$$W = {\cdot}6glA\,;\ k^2 = \frac{15^2}{16}$$

we get

$$l^3 < \frac{7{\cdot}84 \times 10^{11} \times 15^2}{{\cdot}6 \times 981 \times 16}$$

$$l < 2{\cdot}7 \times 10^3 \text{ cm.}$$

Thus the height of the tree cannot exceed about 27 metres.

Determination of Young's Modulus by Flexure.—Young's modulus is often determined by measuring the deflection of a beam supported at both ends and loaded in the middle. If d is the depression of the middle of the bar, then (see p. 93)

$$d = \frac{W}{48qAk^2}\,AB^3$$

where W is the load, AB the length of the bar, q Young's modulus, Ak^2 the moment of inertia of the cross-section of the bar about an axis through the centre of gravity of the section at right angles to the plane of bending.

The value of d can be determined by fixing a needle point to the middle of the bar, and observing through a microscope provided with a micrometer eyepiece the depression of the beam when loaded in the middle with various weights. Another method of measuring d is by means of a very carefully made screw, the end of which is brought into contact with the bar; by measuring the fraction of a turn through which the head of the screw must be turned to renew the contact after the bar has been loaded we can determine the value of d corresponding to given loads. The most accurate method, however, would be an optical one, in which, by Michelson's method, interference fringes are produced by the interference of light reflected from two mirrors, one of which is fixed while the other is attached to the middle point of the bar. By measuring the displacement of the fringes when the load is put on we could determine d, and the method is so delicate that the displacements corresponding to very small loads could be measured.

Another method, due to König, consists in measuring the angle through which the free ends of the bar are bent. The method is represented in Fig. 75. AB is the rod resting on two steel knife edges S_1, S_2. The mirrors P_1, P_2, which are almost at right angles to the rods, are rigidly attached to it. The vertical scale S is reflected first from the mirror P_2, then from the mirror P_1, and then read through the telescope F. The weight is applied to the knife edge r, which is exactly midway between the knife edges S_1, S_2. On looking through the telescope we find one of the divisions of the scale coinciding with the cross wires; on loading the beam another division of the scale will come on the cross wire, and by measuring the distance between these divisions we can determine the angle ϕ through which each free extremity of the bar has been bent. For, let us follow the ray backward from the telescope; when the mirror P_1 is twisted through an angle ϕ, the point where the reflected ray strikes the mirror P_2 is shifted through a distance $2d\phi$, where d is the distance between the mirrors; thus, if the light reflected from P_2 were parallel to its original direction, the scale reading would be altered by $2d\phi$, but the light reflected from P_2 is turned through an angle 4ϕ; this alters the scale reading by $4D\phi$ where D is the distance of the scale S from the mirror P_2, hence v, the total alteration in the scale reading, is given by

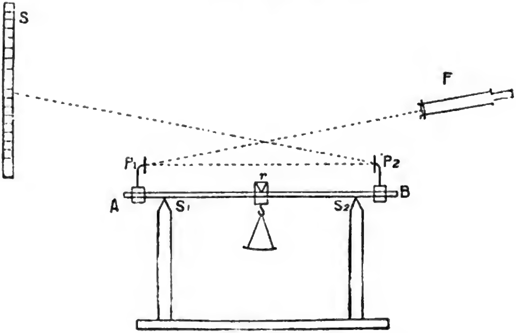

FIG. 75.

$$v = (2d + 4D)\phi$$

Thus
$$\phi = \frac{v}{2d + 4D}$$

but (see p. 91)
$$\phi = \frac{W}{2.qAk^2}\,\frac{l^2}{8}$$

where l is the distance between the knife-edges.

Thus knowing v we can determine q. The advantage of this method is that v, the alteration in the scale reading, may be made very much greater than the depression of the middle of the bar.

The following convenient method for determining both n and q for a wire was given by G. F. C. Searle in the *Philosophical Magazine*, Feb. 1900. AB, CD (Fig. 76) are two equal brass bars of square section, the wire under observation is firmly secured by passing through horizontal holes drilled through the centres G, G′ of the bars. The system can be suspended by two parallel torsionless strings by means of hooks attached to the bars. If now the ends B and D are made to approach each other through equal distances and are then set free the bars will vibrate in a horizontal plane. To a first approximation the centres G and G′ remain at rest, so that the action of the wire on the bar, and therefore of the bar on the wire, is a pure couple; the wire will, therefore, be bent into a horizontal circle and the couple will be qAk^2/ρ. Here q is Young's modulus, Ak^2 the moment of inertia of the cross-section of the wire about an axis through the centre of gravity at right angles to the plane of bending, ρ the radius of curvature of the wire, which is equal to $l/2\phi$ if l is the length of the wire and ϕ the angle through which each bar is twisted. Hence, is K if the moment of inertia of CD about a vertical axis through G, we have

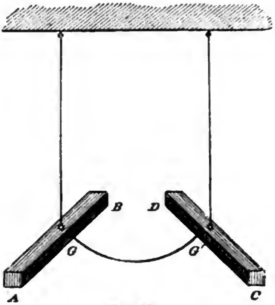

FIG. 76.

$$K\frac{d^2\phi}{dt^2} = -\frac{qAk^2}{\rho} = -\frac{2qAk^2}{l}\phi;$$

hence, if T_1 is the time of vibration,

$$T_1 = 2\pi\sqrt{\frac{Kl}{2.qAk^2}} \qquad (12)$$

The bars are now unhooked from the strings and one clamped to a shelf, so that the wire is vertical; if we make the wire execute torsional vibrations, and T_2 is the time of vibration,

$$T_2 = 2\pi\sqrt{\frac{2Kl}{\pi na^4}} \qquad (13)$$

(see p. 84), n being the coefficient of rigidity and a the radius of the wire As the wire is of circular section,

$$Ak^2 = \frac{\pi a^4}{4};$$

hence by (12) and (13) we have

$$\frac{q}{n} = \frac{T_2^2}{T_1^2}$$

TABLE OF MODULI OF ELASTICITY.

The values of the moduli of elasticity vary so much with the treatment a metal has received in wire-drawing, rolling, annealing, and so on, that whenever they are required for a given specimen it is necessary to determine them, if any degree of accuracy is required. The following table contains the limits within which determinations of the moduli of different metals lie. They are taken from the results of experiments by Wertheim, Kiewiet, Lord Kelvin, Pisati, Baumeister, Mallock, Cornu, Everett, and Katzenelsohn. The values are given in C.G.S. units, n is the rigidity, q Young's modulus, k the bulk modulus, and σ Poisson's ratio.

	$n/10^{11}$	$q/10^{11}$	$k/10^{11}$	σ
Aluminium . .	2·38—3·36	7·4	—	·13
Brass . . .	3·44—4·03	9·48—10·75	10·2—10·85	·226—·469
Copper . . .	3·5—4·5	10·3—12·8	17	·25—·35
Delta-Metal . .	3·6	9·1	10	—
Glass . . .	1·2—2·4	5·4—7·8	3·4—4·2	·20—·26
Gold . . .	3·9—4·2	5·48 (drawn) 8 (rolled)	—	·17
Iron (cast) . .	3·5—5·3	9·8—16	9·7—14·7	·23—·31
Iron (wrought) .	6·6—7·7	17—20	—	—
Lead . . .	·18	·5—1·8	3·7	·375
Phosphor Bronze .	3·6	9·8	—	—
Platinum . .	6·6—7·4	15—17	—	·16
Silver . . .	2·5—2·6	7·0—7·5	—	·37
Steel . . .	7·7—9·8	18—29	14·7—19	·25—·33
Tin . . .	1·5	4·2	—	—
Zinc . . .	3·8	8·7	—	·20

CHAPTER IX.

SPIRAL SPRINGS.

CONTENTS.—Flat Springs—Inclined Springs—Angular deflexion of Free End on Loading—Vibrations of Loaded Spring.

THE theories of bending and twisting have very important applications to the case of spiral springs. By a spiral spring we mean a uniform wire or ribbon wound round a circular cylinder in such a way that the axis of the wire makes a constant angle with the generating lines of the cylinder.

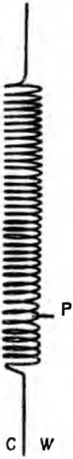

FIG. 77.

The first case we shall consider is that of a spiral spring made of uniform wire of circular cross-section, and wound round the cylinder so that the plane of the wire is everywhere approximately perpendicular to the axis of the cylinder—*i.e.*, a "flat" spring. Let us suppose that such a spring is hung with its axis vertical, and that a weight W, acting along the axis of the cylinder, is applied to an arm attached to the lower end of the spring.

Considering the equilibrium of the portion CP of the spring, the stresses over the cross-section P must be in equilibrium with the force W at C, and hence these stresses must be equivalent to a tangential force W acting upwards, and a couple whose moment is Wa and whose axis coincides with the axis of the wire at P, a being the radius of the cylinder on which the axis of the wire lies. If the diameter of the wire is very small compared with a we may, by the principles explained on p. 89, neglect the effects of the tangential force in comparison with that of the couple and consider the couple alone. This couple is a torsional couple and is constant all along the wire; it will produce, therefore, a uniform rate of twist; if ϕ is the rate of twist, b the radius of the wire, and n its coefficient of rigidity, then we have (see p. 79),

$$Wa = \tfrac{1}{2}\pi n b^4 \phi.$$

Now suppose that we have a series of arms of length a attached to the wire at right angles, the free ends of these arms all being in the axis of the cylinder. Then, if P, Q are two points near together, the effect of the twisting is to increase the vertical distance between the ends of the arms attached to P, Q respectively by PQ × $a\phi$, and since a and ϕ are constants this result will hold whatever the distance between P and Q. Suppose Q is at the fixed and P at the free end of the spring, then the increase in the vertical distance between the arm attached to P and Q will be the vertical depression of the weight W; in this case PQ $= l$, the length of wire in the spring; hence, if d is the depression of W,

$$d = l \times a \times \phi$$

$$= \frac{2lWa^2}{\pi n b^4}$$

Thus d varies directly as the area of the cross-section of the cylinder and inversely as the square of the area of the cross-section of the wire. We see that the depression of the weight is the same as the displacement of the extremity of a horizontal arm of length a attached to the end of the same length of wire when pulled out straight and hung vertically, the end of the horizontal arm being acted on by a horizontal force equal to W at right angles to the arm.

To take a numerical example: suppose we have a steel spring 300 cm. long wound on a cylinder 3 cm. in diameter, the diameter of the wire being ·2 cm.

$$n = 8 \times 10^{11},\ a = 1{\cdot}5,\ b = {\cdot}1.$$

If this spring is loaded with a kilogramme so that $W = 981 \times 10^3$, the depression d will be given by

$$d = \frac{600 \times 981 \times 10^3 \times (1{\cdot}5)^2}{\pi \times 8 \times 10^{11} \times 10^{-4}}$$

$$= 5 \text{ cm. approximately.}$$

Energy in the Spring.—Q, the energy stored in the spring, is (see p. 80) given by the equation

$$Q = \tfrac{1}{4}\pi n l b^4 \phi^2$$

But

$$\phi = \frac{2Wa}{\pi n b^4}$$

thus

$$Q = \frac{W^2 l a^2}{\pi n b^4}$$

$$= \tfrac{1}{2}Wd.$$

This result illustrates the theorem proved on p. 71.

Springs inclined at a finite Angle to the horizontal Plane.—The flat spring, as we have just seen, acts entirely by torsion; in inclined springs however, bending as well as torsion comes into play. Let the axis of the spring make a constant angle α with the horizontal. Let the spring (Fig. 78) be stretched by a weight W acting along the axis of the cylinder on which the spring is wound. Then, considering the equilibrium of the portion AP of the spring, and neglecting as before the tangential stresses at P, we see that the stresses at P must be equivalent to a couple whose moment is Wa, and whose axis is PT, the horizontal tangent to the cylinder at P. This couple may be resolved into two:—one with the moment $Wa\cos\alpha$ and axis along the wire PQ, tending to twist the spring; the second, having the moment $Wa\sin\alpha$ and its axis PN at right angles to the plane of the spring at P tending only to bend the spring. Now the twisting couple $Wa\cos\alpha$ will produce a rate of twist ϕ given by

$$\phi = \frac{Wa\cos\alpha}{nC}$$

where C is a quantity depending on the shape and size of the cross-section of the spring. When the spring is a circular wire of radius b, we have seen that $C = \pi b^4/2$. The couple $Wa\sin a$ will bend the spring and will alter the inclination of the tangents at two neighbouring points PQ by

$$\frac{Wa\sin a \,.\, PQ}{q.D}$$

where $D = Ak^2$, the moment of inertia of the area of the cross-section of the wire of the spring about an axis through its centre of gravity at right angles to the plane of bending.

Let us now consider the effect of these changes on the radial arms which we imagine fixed to the spring. Let us first consider the *vertical* displacements of the ends of the arms at two neighbouring points PQ. Taking first the torsion, the relative motion of the ends is $PQ\phi a$, but in consequence of the inclination of the spring this relative motion is inclined at an angle a with the vertical so that the relative vertical motion is

$$PQa\phi\cos a = \frac{PQ.Wa^2\cos^2 a}{nC}$$

Thus, if l be the length of the wire in the spring, the vertical displacement of the end of the spring due to torsion is

$$\frac{lWa^2\cos^2 a}{nC}$$

Now consider the effect of the bending on the vertical motion of the ends of the rods at PQ. In consequence of the bending, the relative motion is in a plane making an angle a with the horizontal plane and is equal to

$$\frac{Wa\sin a}{qD}PQa$$

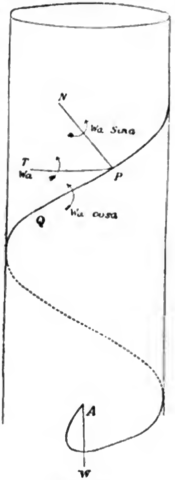

FIG. 78.

To get the vertical component of this we must multiply by $\sin a$, and we see that the vertical displacement due to bending is

$$PQ\frac{Wa^2\sin^2 a}{qD}$$

or for the whole spring

$$\frac{lWa^2\sin^2 a}{qD}$$

Thus the total vertical displacement is

$$lWa^2\left\{\frac{\cos^2 a}{nC} + \frac{\sin^2 a}{qD}\right\}$$

In addition to the vertical displacement there will be an angular displacement of the pointer at the end of the bar which we may calculate as follows. First take the torsion. The arm at P is twisted relatively to the arm at Q through an angle in a plane making an angle $\frac{\pi}{2}-a$ with the horizontal plane equal to PQ × ϕ; the angular motion in the horizontal plane is, therefore,

$$\mathrm{PQ}\times\phi\times\cos\left(\frac{\pi}{2}-a\right)$$

or

$$\mathrm{PQ}\frac{\mathrm{W}a\sin a\cos a}{n\mathrm{C}}$$

And the direction is such that as we proceed along the spring the arms are rotated in the direction in which the spring is wound, so that this angular movement due to the torsion is such as to tend to coil up the spring. The angular deflection due to torsion for the whole spring is, therefore,

$$\frac{l\mathrm{W}a\sin a\cos a}{n\mathrm{C}}$$

Let us now consider the angular deflection due to bending. The arm at P is bent relatively to that at Q through an angle

$$\mathrm{PQ}\frac{\mathrm{W}a\sin a}{q\mathrm{D}}$$

in a plane making an angle a with the horizontal plane; projecting this angle on the horizontal plane the relative angular motion in this plane of the two arms is

$$\mathrm{PQ}\frac{\mathrm{W}a\sin a\cos a}{q\mathrm{D}};$$

thus the angular deflection due to bending for the whole length of the spring is

$$\frac{l\mathrm{W}a\sin a\cos a}{q\mathrm{D}}$$

The deflection in this case is in the opposite direction to that due to the torsion, and is such as to tend to uncoil the spring. The total angular deflection is thus

$$l\mathrm{W}a\sin a\cos a\left\{\frac{1}{n\mathrm{C}}-\frac{1}{q\mathrm{D}}\right\}$$

in the direction tending to coil up the spring. The angular deflection is thus proportional to $\sin a\cos a$ and is greatest when $a=\pi/4$. The deflection tends to coil up the spring or uncoil it according as

$$\frac{1}{n\mathrm{C}}\gtrless\frac{1}{q\mathrm{D}};$$

if the spring is very stiff to resist bending in its own plane, it will coil up under the action of the weight; if, on the other hand, it is very stiff to resist torsion, it will uncoil. This is exemplified by the two springs shown in Figs. 79, 80. The first, which is made of strip metal, with the short dimension in the plane of bending, is very weak to resist bending, and so tends to uncoil when stretched, while the second, which is also made of a strip of metal, but with the long side in the plane of bending, is very stiff to resist bending, and so tends to coil up when stretched. In the case of a circular wire of radius b

$$C = \tfrac{1}{2}\pi b^4$$

$$D = \tfrac{1}{4}\pi b^4;$$

so that

$$\frac{1}{nC} - \frac{1}{qD} = \frac{2}{\pi b^4}\left\{\frac{1}{n} - \frac{2}{q}\right\}$$

For metals q is greater than $2n$, so that

$$\frac{1}{nC} - \frac{1}{qD}$$

is positive, and thus a spring made of circular wire tends to coil up when extended.

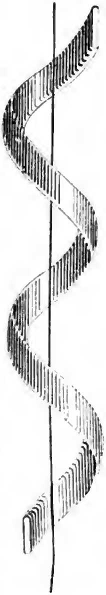

FIG. 79.

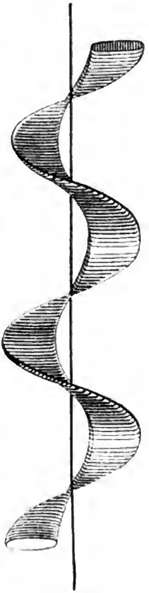

FIG. 80

Vibrations of a Loaded Spring.—We can use the up and down oscillations of a flat spiral spring to determine the coefficient of rigidity of the substance of which the spring is made. Let us take the case of a flat spiral spring made of wire of circular cross-section; then, if the spring is extended a distance x from its position of equilibrium, the potential energy in the spring is (see p. 104) equal to

$$\frac{\pi n b^4}{4 l a^2} x^2 \qquad (1)$$

where n is the coefficient of rigidity, b the radius of cross-section of the wire, a the radius of the cylinder on which the spring is wound, and l the length of the spring. If the end of the spring is loaded with a mass M, the kinetic energy of this mass is equal to

$$\tfrac{1}{2} M \left(\frac{dx}{dt}\right)^2$$

The spring itself is moving up and down, so that there will be some kinetic energy due to the motion of the spring. To a first approximation the vertical motion of a point on the spring is proportional to its distance from the fixed end, so that the velocity at a distance s from the fixed end will be

$$\frac{s}{l} \frac{dx}{dt}$$

If ρ is the mass of unit length of the spring, the mass of an element of length ds is ρds and its kinetic energy is

$$\tfrac{1}{2}\rho\left(\frac{dx}{dt}\right)^2\frac{s^2}{l^2}ds$$

Integrating this expression from $s=o$ to $s=l$, we find that the kinetic energy of the spring is

$$\tfrac{1}{2}\cdot\tfrac{1}{3}\rho l\left(\frac{dx}{dt}\right)^2$$

or if m be the mass of the spring

$$\tfrac{1}{2}\frac{m}{3}\left(\frac{dx}{dt}\right)^2$$

hence the total kinetic energy is equal to

$$\tfrac{1}{2}\left\{\mathrm{M}+\frac{m}{3}\right\}\left(\frac{dx}{dt}\right)^2$$

Since the sum of the kinetic and potential energy is constant

$$\tfrac{1}{2}\left(\mathrm{M}+\frac{m}{3}\right)\left(\frac{dx}{dt}\right)^2+\frac{\pi nb^4x^2}{4la^2}$$

is constant, hence differentiating with respect to t we have

$$\left(\mathrm{M}+\frac{m}{3}\right)\frac{d^2x}{dt^2}+\frac{\pi nb^4}{2la^2}x=0$$

This equation represents a periodic motion, the time T of a complete vibration being given by the equation

$$\mathrm{T}=2\pi\sqrt{\frac{\mathrm{M}+m/3}{\pi nb^4/2la^2}}$$

When T has been determined n can be found by this equation.

Angular Oscillations.[*]—We can prove in a similar way that T, the time of vibration of a suspended bar about the vertical axis, is given by the equation

$$\mathrm{T}_1=2\pi\sqrt{\frac{\mathrm{M}k^2+ma^2/3}{\pi qb^4/4}}$$

where $\mathrm{M}k^2$ is the moment of inertia of the bar about the vertical axis and q Young's modulus for the wire, by measuring T, we can determine q.

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[*] Ayrton and Perry, *Proc. R.S.*, vol. xxxvi., p. 311; Wilberforce, *Phil. Mag.*, Oct. 1894.

CHAPTER X.

IMPACT.

CONTENTS.—Co-efficient of Restitution—Newton's Experiments—Hodgkinson's Experiments—Example of Collision of Railway Carriages—Hertz's Investigations—Table of Co-efficients.

Co-efficient of Restitution.—An interesting class of phenomena depending on the elasticity of matter is that of collision between elastic bodies. The laws governing these collisions were investigated by Newton and his contemporaries, who used the following method. The colliding bodies were spherical balls suspended by strings in the way shown in Fig. 81; the balls, after falling from given heights, struck against each other at the lowest point, and after rebounding again reached a certain height. By measuring these heights (and allowing, as Newton did, for the resistance of the air) the velocities of the balls before and after collision can be determined. Newton in this way showed that when the collision was direct—*i.e.*, when the relative velocities of the two bodies at the instant of collision was along the common normal at the point of impact—the relative velocity after impact bore a constant ratio to the relative velocity before impact — the relative velocity being, of course, reversed in direction. Thus, if u, v are the velocities of the bodies before impact, u being the velocity of the more slowly moving body, while U, V are the velocities after impact, then

$$U - V = e\,(v - u) \qquad (1)$$

where e is a quantity called the coefficient of restitution, and Newton's experiments showed that e depended only on the materials of which the balls were made, and not on the masses or relative velocities. A series of experiments were made by Hodgkinson, the results of which were in general agreement with Newton's. Hodgkinson found, however (*Report of British Association*, 1834), that when the initial relative velocity was very large e was smaller than it was with moderate velocity.

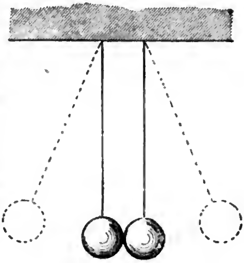

FIG. 81.

Vincent* has shown that the coefficient of restitution is given by the equation $e = e_0 - bu$, where u is the velocity of approach and e_0 and b are constants.

Equation (1) and the equation

$$mu + Mv = mU + MV \qquad (2)$$

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* Vincent, *Proceedings Cambridge Philosophical Society*, vol. x p. 332.

which expresses that the momentum of the system of two bodies is not altered by the impact, m and M being the masses of the bodies, are sufficient to determine U, V; solving equations (1) and (2) we find

$$U = \frac{mu + Mv}{m + M} + e\frac{M}{m + M}(v - u)$$

$$V = \frac{mu + Mv}{m + M} - e\frac{m}{m + M}(v - u)$$

Hence we have

$$\tfrac{1}{2}mU^2 + \tfrac{1}{2}MV^2 = \tfrac{1}{2}mu^2 + \tfrac{1}{2}Mv^2 - \tfrac{1}{2}(1 - e^2)\frac{Mm}{M + m}(v - u)^2 \qquad (3)$$

Thus the kinetic energy after impact is less than the kinetic energy before impact by

$$\tfrac{1}{2}(1 - e^2)\frac{Mm}{M + m}(v - u)^2 \qquad (4)$$

Thus, if e is unity there is no loss of kinetic energy. In all other cases there is a finite loss of kinetic energy, some of it being transformed during the collision into heat; a small part only of it may in some cases be spent in throwing the balls into vibration about their figures of equilibrium.

Collision of Railway Carriages.—To get a clearer idea of what goes on when two elastic balls impinge against each other, let us take the case of a collision between two railway carriages running on frictionless rails, each carriage being provided with a buffer spring. When the carriages come into collision, the first effect is to compress the springs, the pressure which one spring exerts on another is transmitted to the carriages, and the momentum of the carriage that was overtaken increases, while that of the other diminishes; this goes on until the two carriages are moving with the same velocity, when the springs have their maximum compression and the pressure between them is a maximum. The kinetic energy of the carriages is now less than it was before impact by

$$\tfrac{1}{2}\frac{Mm}{M + m}(v - u)^2$$

and this energy is stored in the springs. The springs having reached their maximum compression begin to expand, increasing still further the momentum of the front carriage and diminishing that of the carriage in the rear. This goes on until the springs have regained their original length, when the pressure between them vanishes and the carriages separate. There is now no strain energy in the springs, and the kinetic energy in the carriages after the collision has ceased is the same as it was before it began.

The reader who is acquainted with the elements of the differential calculus will find it advantageous to consider the analytical solution of the problem, which is very simple. Let x, y be the co ordinates of the centres of gravity of the first and second carriages respectively, μ, μ' the strength of the springs attached to these carriages (by the strength of a spring we mean the force required to produce unit extension of the spring), ξ, η the compressions of these springs, and P the pressure between them; then we have

$$m\frac{d^2x}{dt^2} = \mathrm{P}, \quad \frac{\mathrm{M}d^2y}{dt^2} = -\mathrm{P}$$

$$\mu\xi = \mathrm{P}, \qquad \mu'\eta = \mathrm{P}$$

$$x - y = \text{constant} - (\xi + \eta)$$

The solution of these equations is

$$\mathrm{P} = (v-u)\omega\frac{\mathrm{M}m}{\mathrm{M}+m}\sin\omega t$$

where $\omega = \sqrt{\frac{\mu\mu'}{\mu+\mu'}\frac{\mathrm{M}+m}{\mathrm{M}m}}$; u and v are the initial velocities of the carriages, and t is measured from the instant when the collision began.

$$m\frac{dx}{dt} = \frac{m}{\mathrm{M}+m}\{mu + \mathrm{M}v\} - \frac{\mathrm{M}m}{\mathrm{M}+m}(v-u)\cos\omega t$$

$$\mathrm{M}\frac{dy}{dt} = \frac{\mathrm{M}}{\mathrm{M}+m}\{mu + \mathrm{M}v\} + \frac{\mathrm{M}m}{\mathrm{M}+m}(v-u)\cos\omega t$$

Thus the springs have their maximum compression when $\frac{dx}{dt} = \frac{dy}{dt}$, *i.e.*, when $\omega t = \pi/2$, or $t = \frac{\pi}{2\omega}$; at this instant the energy stored in the first spring

$$= \tfrac{1}{2}\mathrm{P}\xi$$

$$= \tfrac{1}{2}\frac{\mathrm{P}^2}{\mu} = \tfrac{1}{2}(v-u)^2\frac{\mu'}{\mu+\mu'}\,\frac{\mathrm{M}m}{\mathrm{M}+m}$$

while the energy in the second spring is equal to

$$\tfrac{1}{2}\mathrm{P}\eta$$

$$= \tfrac{1}{2}\frac{\mathrm{P}^2}{\mu'} = \tfrac{1}{2}(v-u)^2\frac{\mu}{\mu+\mu'}\,\frac{\mathrm{M}m}{\mathrm{M}+m}$$

At the instant of greatest compression the amounts of energy stored in the two springs are inversely as the strengths of the springs.

The springs regain their original length and the collision ceases when $\mathrm{P} = 0$—*i.e.*, when $\omega t = \pi$, or

$$t = \frac{\pi}{\omega} = \pi\sqrt{\frac{\mathrm{M}m}{\mathrm{M}+m}\,\frac{\mu+\mu'}{\mu\mu'}}$$

this is the time the collision lasts. We see that it increases as the masses of the carriages increase and diminishes as the strengths of the springs increase. It is independent of the relative velocity of the carriages before impact.

In the case of the collision between elastic bodies the elasticity of the material serves instead of the springs in the preceding example. The bodies when they come into collision flatten at the point of contact so that the bodies have a finite area in common. In the neighbourhood of this area each body is compressed; the compression attains a maximum, then diminishes and vanishes when the bodies separate. The theory of the collision between elastic bodies has been worked out from this point of view by Hertz (*see* Collected Papers, English Translation, p. 146), who finds expressions for the area of the surface in contact between the colliding bodies, the duration of the contact and the maximum pressure. The duration of contact of two equal spheres was proved by Hertz to be equal to

$$2{\cdot}9432\mathrm{R}\sqrt[5]{\frac{25\pi^2 s^2(1-\sigma^2)^2}{8(v-u)q^2}}$$

where R is the radius of either of the spheres, s the density of the sphere, q and σ respectively Young's modulus and Poisson's ratio for the substance of which the spheres are made. Hamburger has measured the time two spheres are in contact by making the spheres close an electric circuit whilst they are in contact and measuring the time the current is flowing. The results of his experiments are given in the following table. They relate to the collision of brass spheres 1·3 cm. in radius:

Relative Velocity in cm. per sec.	7·37	12·29	19·21	29·5
Duration of collision (calculated)	·000185	·000167	·000153	·000140
" " " (observed) .	·000196	·000173	·000157	·000148

The duration of the impact is several times the gravest time of vibration of the body. In order to start such vibrations with any vigour the time of collision would have to be small compared with the time of vibration. We conclude that only a small part of the energy is spent in setting the spheres in vibration.

As an example of the order of magnitude of the quantities involved in the collision of spheres we quote the results given by Hertz for two steel spheres 2·5 cm. in radius meeting with a relative velocity of 1 cm. per second. The radius of the surface of contact is ·013 cm. The time of contact is ·00038 seconds. The maximum total pressure is 2·47 kilogrammes and the maximum pressure per unit area is 7300 kilogrammes per square centimetre.

In this theory and in the example of the carriages with springs we have supposed that the work done on the springs is all stored up as available potential energy and is ultimately reconverted into kinetic energy, so that the total kinetic energy at the end of the impact is the same as at the beginning. This is the case of the impact of what are called perfectly elastic bodies, for which the co-efficient of restitution is equal to unity. In other cases we see by equation (3) that, instead of the whole work done on the springs being reconverted into kinetic energy, only the constant fraction e^2 of it is so reconverted, the rest being ultimately converted into heat. Now our study of the elastic properties of bodies has shown many examples in which it is impossible to convert the energy due to strain into kinetic energy and the kinetic energy back again into energy due to strain without dissipation. We may mention the phenomena of elastic fatigue or viscosity of metals (see page 57), as exemplified by the torsional vibrations of a metal wire, where the successive transformations of the energy were accompanied by a continued loss of available energy. Again, the elastic after-effect would prevent a total conversion of strain energy into mechanical energy. For example, if we load a wire up to a certain point, and measure the extension corresponding to any load, then gradually unload the wire, if the straining has gone beyond the elastic limit the extensions during unloading will not be the same as during loading; and in this case there will in any complete cycle be a loss of mechanical energy proportional to the area included between the curves for loading and unloading. The percentage loss in this case would depend upon the intensity of the maximum stress; if this did not strain the body beyond its elastic limit there would be no loss from this cause, while if the maximum strain exceeded this limit the loss might be considerable. This may be the reason why the value of e diminishes as the relative velocity at the moment of collision increases, for Hertz has shown that the maximum pressure increases with the relative velocity being proportional to the 2/5ths power of the velocity, while it is independent of the size of the balls. Thus the greater the relative velocity the more will the maximum pressure exceed the elastic limit and the larger the amount of heat produced. In addition to the loss of energy by the viscosity of metals and hysteresis there is in many cases of collision permanent deformation of the surface in the neighbourhood of the surface of contact. This is very evident in the case of lead and brass. The harder the body the greater the value of e. We can see the reason for this if we remember that the hardness of a body is measured by the maximum stress it can suffer without being strained beyond the elastic limit, while any strain beyond the elastic limit would increase the amount of heat produced and so diminish the value of e.

When we consider the various ways in which imperfections in the elastic property can prevent the complete transformation of the energy due to strain into kinetic energy and *vice versâ*, it is somewhat surprising that the laws of the collision of imperfectly elastic bodies are as simple as Newton's and Hodgkinson's experiments show them to be, for these laws express the fact that in the collision a constant fraction, e^2, of the initial kinetic energy is converted into heat, and that this fraction is independent of the size of the spheres and only varies very slowly with the relative velocity at impact. For example, Hodgkinson's experiments show that when the relative velocity at impact was increased threefold the value of e in the case of the collision between cast-iron spheres only diminished from ·69 to ·59. A series of experiments on the impact of bodies meeting with very small relative velocities would be very interesting, for with small velocities the stresses would diminish, and if these did not exceed those corresponding to elastic limits some of the causes of the dissipation of energy would be eliminated, and it is possible that the value of e might be considerably increased.

We find, too, from experiment that bodies require time to recover even from small strain, so that, if the rise and fall of the stress is very rapid, there may be dissipation of energy in cases where the elastic limit for slowly varying forces is not overstepped.

Hodgkinson gives the following formula for the value of e_{AB}, when two different bodies A and B collide, in terms of the values of e_{AA} for the collision between two bodies each of material A and e_{BB}, the value for the collision between two bodies each of material B.

$$e_{AB} = \frac{\dfrac{e_{AA}}{q_1} + \dfrac{e_{BB}}{q_2}}{\dfrac{1}{q_1} + \dfrac{1}{q_2}}$$

and he finds this formula agrees well with his experiments.

The following considerations would lead to a formula giving e_{AB} in terms of e_{AA} and e_{BB}. Hertz has shown that the displacements of the bodies A and B in the direction of the common normal to the two surfaces over which the bodies touch are proportional to

$$\frac{1-\sigma_1^2}{q_1} \text{ and } \frac{1-\sigma_2^2}{q_2}$$

where σ_1, σ_2 are the values of Poisson's ratio for the bodies A and B and q_1, q_2 the values of Young's modulus. Now the stresses are equal, so that, assuming that the quantities of work done on the two bodies are in the ratio of the displacements, then, if E is the whole work done,

$$\frac{\dfrac{1-\sigma_1^2}{q_1}E}{\dfrac{1-\sigma_1^2}{q_1} + \dfrac{1-\sigma_2^2}{q_2}} \text{ and } \frac{\dfrac{1-\sigma_2^2}{q_2}E}{\dfrac{1-\sigma_1^2}{q_1} + \dfrac{1-\sigma_2^2}{q_2}}$$

will be the amounts done on the two bodies. Now the first body converts $1 - e^2_{AA}$ and the second $1 - e^2_{BB}$ of this work into heat; hence the energy converted into heat will be

$$\frac{(1-e^2_{AA})\dfrac{1-\sigma_1^2}{q_1} + (1-e^2_{BB})\dfrac{1-\sigma_2^2}{q_2}}{\dfrac{1-\sigma_1^2}{q_1} + \dfrac{1-\sigma_2^2}{q_2}} E$$

and this must equal

$$(1-e^2_{AB})E$$

hence

$$e^2_{AB} = \frac{e^2_{AA}\dfrac{1-\sigma_1^2}{q_1} + e^2_{BB}\dfrac{1-\sigma_2^2}{q_2}}{\dfrac{1-\sigma_1^2}{q_1} + \dfrac{1-\sigma_2^2}{q_2}}$$

The following table of the values of e is taken from Hodgkinson's *Report to the British Association*, 1834:

Cast-iron balls	·66	Clay	·17
Cast-iron—lead	·13	Clay—soft brass	·16
Cast-iron—boulder stone	·71	Glass	·94
Boulder stone—brass	·62	Cork	·65
Boulder stone—lead	·17	Ivory	·81
Boulder stone—elm	·56	Lead—glass	·25
Elm balls	·60	Soft brass—glass	·78
Soft brass (16 pt. Cu. and 1 pt. tin)	·36	Bell metal—glass	·87
Bell metal (16 pt. Cu. and 4 pt. tin)	·59	Cast-iron—glass	·91
Lead	·20	Lead—ivory	·44
Lead—elm	·41	Soft brass—ivory	·78
Elm—soft brass	·52	Bell metal—ivory	·77

The case where a permanent deformation is produced has recently been investigated by Vincent (*Proceedings Cambridge Philosophical Society*, vol. x. p. 332). The case taken is that of the indentation produced in lead or paraffin by the impact of a steel sphere. He finds that the volume of the dent is proportional to the energy of the sphere just before impact; that during the impact (*i.e.*, while the lead is flowing) the pressure between the sphere and the lead is constant and varies from 6×10^8 to 13×10^8 dynes per square centimetre for different specimens of lead; for paraffin the corresponding pressure is about 10^7 dynes per square centimetre.

CHAPTER XI.

COMPRESSIBILITY OF LIQUIDS.

CONTENTS.—Changes in Volume of a Tube under Internal and External Pressure—Measurements of Compressibility of Liquids by methods of Jamin, Regnault, Buchanan and Tait, Amagat—Compressibility of Water—Effects of Temperature and Pressure—Compressibility of Mercury and other Liquids—Tensile Strength of Liquids.

THE fact that water is compressible under pressure was established in 1762 by Canton, and since then measurements of the changes of volume of liquids under pressure have been made by many physicists.

The problem is one beset with experimental difficulties, some of which may be illustrated by considering the case of a liquid inclosed in a vessel such as a thermometer; when pressure is applied to the liquid, the depression of the liquid in the stem will be due partly to the contraction of the liquid under pressure and partly to the expansion of the bulb of the thermometer. In order, then, to be able to determine from the depression of the liquid the compressibility of water we must be able to estimate the alteration in volume of the tube under pressure. We shall therefore consider in some detail the alteration in volume of a vessel subject to internal and external pressure. We shall take the case of a long cylindrical tube with flat ends exposed to an external pressure p_1 and an internal pressure p_0. The strain in such a cylinder has been shown by Lamé to be (1) a radial displacement ρ given by the equation

$$\rho = \mathrm{A}r + \frac{\mathrm{B}}{r}$$

where r is the distance of the point under consideration from the axis of the cylinder and A and B constants, and (2) an extension parallel to the axis of the cylinder.

The radial displacement ρ involves an elongation along the radius equal to $d\rho/dr$ and an elongation at right angles to ρ in the plane at right angles to the axis of the cylinder equal to ρ/r. Let the elongations along the radius, at right angles to it and to the axis of the cylinder, and along the axis be denoted by e, f, g respectively, and let P, Q, R be the normal stresses in these directions; then by equation (1), p. 72, we can easily prove

$$\left.\begin{aligned} \mathrm{P} &= \left(k + \frac{4n}{3}\right)e + \left(k - \frac{2n}{3}\right)(f+g) \\ \mathrm{Q} &= \left(k + \frac{4n}{3}\right)f + \left(k - \frac{2n}{3}\right)(g+e) \\ \mathrm{R} &= \left(k + \frac{4n}{3}\right)g + \left(k - \frac{2n}{3}\right)(e+f) \end{aligned}\right\} \qquad (1)$$

where k is the bulk modulus and n the coefficient of rigidity.

Since
$$e=\frac{d\rho}{dr} \text{ and } f=\frac{\rho}{r}$$

we have
$$e=\mathrm{A}-\frac{\mathrm{B}}{r^2}, \qquad f=\mathrm{A}+\frac{\mathrm{B}}{r^2}$$

Thus the radial stress is equal to

$$2k\mathrm{A}+\frac{2n}{3}\left(\mathrm{A}-\frac{3\mathrm{B}}{r^2}\right)+\left(k-\frac{2n}{3}\right)g$$

If a and b are respectively the internal and external radii of the tube, then when $r=a$ the radial stress is equal to $-p_0$ and when $r=b$ the radial stress is equal to $-p_1$, hence we have

$$-p_0=2k\mathrm{A}+\frac{2n}{3}\left(\mathrm{A}-\frac{3\mathrm{B}}{a^2}\right)+\left(k-\frac{2n}{3}\right)g \qquad (2)$$

$$-p_1=2k\mathrm{A}+\frac{2n}{3}\left(\mathrm{A}-\frac{3\mathrm{B}}{b^2}\right)+\left(k-\frac{2n}{3}\right)g \qquad (3)$$

The whole force parallel to the axis tending to stretch the cylinder is

$$\pi a^2 p_0-\pi b^2 p_1$$

hence the stress in this direction is equal to

$$\frac{\pi a^2 p_0-\pi b^2 p_1}{\pi(b^2-a^2)}$$

The stress parallel to the axis is, however, equal to

$$\left(k+\frac{4n}{3}\right)g+\left(k-\frac{2n}{3}\right)2\mathrm{A}$$

hence we have

$$\frac{a^2 p_0-b^2 p_1}{b^2-a^2}=\left(k+\frac{4n}{3}\right)g+\left(k-\frac{2n}{3}\right)2\mathrm{A} \qquad (4)$$

From (2), (3) and (4) we get

$$\mathrm{A}=g=\frac{1}{3k}\,\frac{(p_0a^2-p_1b^2)}{b^2-a^2}$$

and
$$\mathrm{B}=\frac{1}{2n}\,\frac{a^2b^2}{(b^2-a^2)}(p_0-p_1)$$

Since the radial displacement is $\mathrm{A}r+\frac{\mathrm{B}}{r}$, the internal volume of the tube when strained is $\pi\left(a+\mathrm{A}a+\frac{\mathrm{B}}{a}\right)^2 l(1+g)$

where l is the length of the tube; hence, retaining only the first powers of the small quantities A, B and g, we have, if δv_1 is the change in the internal volume,

$$\delta v_1=\pi a^2 l\left\{\frac{p_0a^2-p_1b^2}{(b^2-a^2)}\,\frac{1}{k}+\frac{b^2}{b^2-a^2}\,\frac{p_0-p_1}{n}\right\}$$

and if δv_2 is the change in the external volume,

$$\delta v_2 = \pi b^2 l \left\{ \frac{p_0 a^2 - p_1 b^2}{b^2 - a^2} \frac{1}{k} + \frac{a^2}{b^2 - a^2} \frac{p_0 - p_1}{n} \right\}$$

Methods of Measuring Compressibility of Liquids.—There are two cases of special importance in the determination of the compressibility of fluids: the first is when the internal and external pressures are equal; in this case $p_0 = p_1$, and we have

$$\delta v_1 = -\frac{\pi a^2 l}{k} p_0$$

Thus the diminution of the volume is independent of the thickness of the walls of the tube. Some experimenters have been led into error by supposing that, if the walls of the tube were very thin, there would be no appreciable diminution in the volume of the tube. If the vessel had been filled with liquid which was subject to the pressure p_0, the diminution in the volume of the liquid would be $\pi a^2 l p_0 / \mathrm{K}$, where K is the bulk modulus of the liquid. The diminution of volume of the liquid minus that of the vessel is therefore

$$\pi a^2 l p_0 \left(\frac{1}{\mathrm{K}} - \frac{1}{k} \right)$$

thus by experiments with equal pressures inside and out, which was Regnault's method, we determine

$$\frac{1}{\mathrm{K}} - \frac{1}{k}$$

so that to deduce K we must know k.

Another method, used by Jamin, was to use internal pressure only, when the apparent change in the volume of the liquid is the sum of the changes of volumes of the liquid and of the inside of the vessel. Jamin thought that he determined the change of volume of the vessel by placing it in an outer vessel full of water and measuring the rise of the water in a graduated capillary tube attached to this outer vessel; by subtracting this change in volume from the apparent change he thought he got the change in volume of the liquid without requiring the values of the elastic constants of the material of which the vessel is made. A little consideration will show, however, that this is not the case. Let δv be the change in the volume of the liquid, δv_1 the change in the internal volume, δv_2 that in the external volume; it is δv_2 that is measured by the rise of liquid in the capillary tube attached to the vessel containing the tube in which the liquid is compressed.

Observations on the liquid inside the tube give

$$\delta v + \delta v_1$$

if we subtract Jamin's correction we get

$$\delta v + \delta v_1 - \delta v_2$$

substituting the values of δv_1 and δv_2 when $p_1 = o$ we find

$$\delta v + \delta v_1 - \delta v_2 = \delta v - \frac{\pi a^2 l p_0}{k} \text{ and } \delta v = \frac{\pi a^2 l p_0}{\mathrm{K}}$$

Hence, after applying Jamin's correction, we get $\pi a^2 l p_0\left(\frac{1}{\mathrm{K}} - \frac{1}{k}\right)$ the same quantity as was determined by Regnault's method, so that to get K by Jamin's method we require to know k.

The apparatus used by Regnault in his experiments on the compressibility of liquids (*Mémoires de l'Institut de France*, vol. xxi. p. 429) was similar to that represented in Fig. 82. The piezometer was filled with the liquid whose compressibility was to be measured, the greatest care being taken to get rid of air-bubbles. The liquid reached up into the graduated stem of the piezometer, the volume between successive marks on the stem being accurately known. The piezometer was placed in an outer vessel which was filled with water and the whole system placed in a large tank filled with water, the object being to keep the temperature of the system constant. The tubes shown in the system were connected with a vessel full of compressed air, the pressure of which was measured by a carefully tested manometer; the tubes were so arranged that by turning on the proper taps pressure could be applied (1) to the outside of the piezometer and not to the inside; (2) simultaneously to the outside and the inside; (3) to the inside and not to the outside. The piezometer used by Regnault was in the form of a cylindrical tube with hemispherical ends. For simplicity let us take the case (represented in the figure) of a piezometer in the form of a cylinder with flat ends, to which the foregoing investigation applies. If ω_1, ω_2, ω_3 are the apparent diminution in the volume of the liquid in the three cases respectively, the pressure being the same, we have by the preceding theory

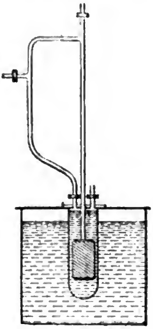

Fig. 82.

$$\omega_1 = -\frac{\pi a^2 b^2 l}{b^2 - a^2} p\left(\frac{1}{k} + \frac{1}{n}\right)$$

$$\omega_2 = \pi a^2 l p\left(\frac{1}{\mathrm{K}} - \frac{1}{k}\right)$$

$$\omega_3 = \frac{\pi a^2 l}{b^2 - a^2} p\left\{\frac{b^2 - a^2}{\mathrm{K}} + \frac{a^2}{k} + \frac{b^2}{n}\right\}$$

Hence $$\omega_1 + \omega_3 = \omega_2$$

a relation by which we can check to some extent the validity of the theoretical investigation. Such a check is very desirable, as in this investigation we have assumed that the material of which the piezometer is made is isotropic and that the walls of the piezometer are of uniform thickness, conditions which are very difficult to fulfil, while it is important to ensure that a failure in any one of them has not been sufficient to appreciably impair the accuracy of the theoretical investigations. Regnault in his investigations adopted Lamé's assumption that Poisson's ratio is equal to 1/4; on this assumption $n = \frac{3}{5}k$, so that the measurement of ω_1

gives the value of k, and then the measurement of ω_2 the value of K, the bulk modulus for the liquid. This was the method adopted by Regnault. It is, however, open to objection. In the first place, the determinations which have been made of the value of Poisson's ratio for glass range from ·33 to ·22, instead of the assumed value ·25, while, secondly, the equation by which k is determined from measurements of ω_1 is obtained on the assumption of perfect uniformity in the material which it is difficult to verify. It is thus desirable to determine k for the material of which the piezometer is made by a separate investigation, and then to determine the compressibility of the liquids by using the simplest relation obtained between the apparent change in volume of the liquid and the pressure; this is when the inside and outside of the piezometer are exposed to equal pressures. The most direct, and probably the most accurate, way of finding k for a solid is to measure the longitudinal contraction under pressure. An arrangement which enables this to be done with great accuracy is described by Amagat in the *Journal de Physique*, Series 2, vol. viii. p. 359. The method was first used by Buchanan and Tait. Another method of determining k for a solid is to make a tube of the solid closed by a graduated capillary tube as in Fig. 83. The tube and part of the capillary being filled with water, a tension P is applied to the tube, the tube stretches and the internal volume increases, the increase in volume being measured by the descent of the liquid in the capillary tube; if v is the original internal volume, δv the increase in this volume, then we see by the investigation, p. 72, that

$$\frac{\delta v}{v} = \frac{P}{3k}$$

If we have found k, then K can be found by means of the piezometer.

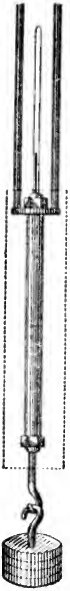

Fig. 83.

If we can regard the compressibility of any liquid, say mercury, as known, the most accurate way of finding the compressibility of any other liquid would be to fill the piezometer first with mercury, and determine the apparent change of volume when the inside and outside of the piezometer are exposed to the same pressure; then fill the piezometer with the liquid and again find the apparent change in volume. We shall thus get two equations from which we can find the value of K for the liquid and k for the piezometer.

Results of Experiments.—The results of experiments made by different observers on the compressibility of water are given below.

Regnault.*—Temperature not specified; pressures from 1 to 10 atmospheres—

compressibility per atmosphere = 0.000048.

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* *Mémoires de l'Institut de France*, vol. xxi. p. 429.

GRASSI.*		PAGLIANI and VICENTINI.†		RONTGEN and SCHNEIDER.‡	
Temp.	Compressibility per atmosphere.	Temp.	Compressibility per atmosphere.	Temp.	Compressibility per atmosphere.
0·0	503×10^{-7}	0.0	503×10^{-7}	0·0	512×10^{-7}
1·5	515	2·4	496	9·0	481
4·0 max. density pt.	499	15·9	450	18·0	462
		49·3	403		
10·8	480	61·1	389		
13·4	477	66·2	389		
18·0	462	77·4	398		
25.0	456	99.2	409		
34·5	453				
43 0	442				
53·0	441				

Tait§ has found that the effect of temperature and pressure, for temperatures between 6° C. and 15° C. to pressures from 150 to 500 atmospheres, may be represented by the empirical formula

$$\frac{v_0 - v}{pv_0} = 0·0000489 - 0·00000025t - 0.0000000067p$$

where v is the volume at t^0 C. under the pressure of p atmospheres and v_0 the volume at t^0 under one atmosphere. Thus the compressibility diminishes as the pressure increases.

The numbers given above, from Grassi's experiments, indicate that water has a maximum compressibility at a temperature between 0° and 4° C.: this result has not, however, been confirmed by subsequent observers. The results of Pagliani and Vicentini indicate a *minimum* compressibility at a temperature between 60° and 70° C.

The results of various observers on the compressibility of mercury are given in the following table:

Observer.	Compressibility per atmosphere.
Colladon and Sturm‖	$35·2 \times 10^{-7}$
Aimé¶	$39·0 \times 10^{-7}$
Regnault**	$35·2 \times 10^{-7}$
Amaury and Descamps††	$38·6 \times 10^{-7}$
Tait‡‡	$36·0 \times 10^{-7}$
Amagat§§	$39·0 \times 10^{-7}$
De Metz‖‖	$37·4 \times 10^{-7}$
Mean	$37·9 \times 10^{-7}$

The compressibility of mercury, like that of most fluids, increases as the temperature increases. According to De Metz, the compressibility at t° C. is given by

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* Grassi, *Annales de Chimie et de Physique* [3], 31, p. 437, 1851.

† Pagliani and Vicentini, *Nuovo Cimento* [3], 16, p. 27, 1884.

‡ Röntgen and Schneider, *Wied. Ann.*, 33, p. 644, 1888.

§ Tait, *Properties of Matter*, 1st ed. (1885), p. 190.

‖ Colladon and Sturm, *Ann. de Chimie et de Physique*, 36, p. 137, 1827.

¶ Aimé, *Annales de Chimie et de Physique* [3], 8, p. 268, 1843.

** Regnault, *Mémoires de l'Institut de France*, 21, p. 429, 1847.

†† Amaury and Descamps, *Compt. Rend.*, 68, p. 1564, 1869.

‡‡ Tait, *Challenger Report*, vol. ii. part iv.

§§ Amagat, *Journal de Physique* [2], 8, p. 203, 1889.

‖‖ De Metz, *Wied. Ann.*, 47, p. 731, 1892.

$$37{\cdot}4 \times 10^{-7} + 87{\cdot}7 \times 10^{-10}t$$

The compressibilities of a number of liquids of frequent occurrence are given below.

Fluid.	Compressibility per atmosphere.	Temp.	Observer.
Sea-water	436×10^{-7}	17·5°	Grassi
Ether	1156×10^{-7}	0°	Quincke
"	1110×10^{-7}	0°	Grassi
Alcohol	828×10^{-7}	0°	Quincke
"	959×10^{-7}	17·5°	"
"	828×10^{-7}	7·3°	Grassi
Methyl alcohol	913×10^{-7}	13·5°	"
Turpentine	582×10^{-7}	0°	Quincke
"	779×10^{-7}	18·6	"
Chloroform	625×10^{-7}	8·5°	Grassi
Glycerine	252×10^{-7}	0°	Quincke
Olive oil	486×10^{-7}	0°	"
Carbon bisulphide	539×10^{-7}	0°	"
" "	638×10^{-7}	17°	"
Petroleum	650×10^{-7}	0°	"
"	745×10^{-7}	19·2°	"

Quincke's paper is in *Wiedemann's Annalen*, 19, p. 401, 1883. References to the papers by the other observers have already been given. An extensive series of investigations on the compressibility of solutions has been made by Röntgen and Schneider (*Wied. Ann.*, 29, p. 165, and 31, p. 1000), who have shown that the compressibility of aqueous solutions is less than that of water. For the details of their results we must refer the reader to their paper.

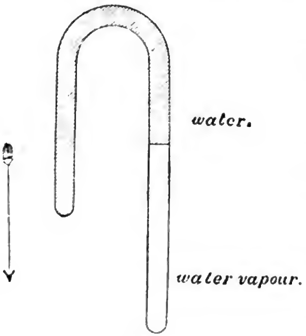

Fig. 84.

Tensile Strength of Liquids.—Liquids from which the air has been carefully expelled can sustain a considerable pull without rupture. The best known illustration of this is the sticking of the mercury at the top of a barometer-tube. If a barometer-tube filled with mercury be carefully tilted up to a vertical position, the mercury sometimes adheres to the top of the tube, and the tube remains filled with mercury, although the length of the column is greater than that which the normal barometric pressure would support, and the extra length of mercury is in a state of tension. Another method of showing that liquids can sustain tension without rupture is to use a tube like that in Fig. 84, filled with water and the vapour of water, and from which the air has been carefully expelled by boiling the water and driving the air out by the steam.* If the water occupies the position indicated in the figure, the tube mounted on a board may be moved rapidly forward in the direction of the arrow, and then brought suddenly to rest by striking the board against a table without the water column breaking, although the column must have experienced a considerable impulsive tension. If the column does break, a small bubble of air can generally be observed at the place of rupture, and until this bubble has been removed the column will break with great ease. On the removal of the bubble by tapping, the column can again sustain a considerable shock without rupture.

Professor Osborne Reynolds used the following method for measuring the tension liquids would stand without breaking. ABCD, Fig. 85, is a glass U-tube, closed at both ends, containing air-free liquid ABC and vapour of the liquid CD. The tube is fixed to a board and whirled by a lathe about an axis O a little beyond the end A and perpendicular to the plane of the board. If CE is an arc of an circle with centre O, then when the board is rotating the liquid EA is in a state of tension, the tension increasing from E to A, and being easily calculable if we know the velocity of rotation. By this method Professor Osborne Reynolds found that water could sustain a tension of 72·5 pounds to the square inch without rupture, and Professor Worthington, using the same method, found that alcohol could sustain 116 and strong sulphuric acid 173 pounds per square inch. This method measures the stress liquids can sustain without rupture. Berthelot has used a method by which the strain is measured. The liquid freed from air by long boiling nearly filled a straight thick-walled glass tube, the rest of the space being occupied by the vapour of the liquid. The liquid was slightly heated until it occupied the whole tube; on cooling, the liquid continued for some time to fill the tube, finally breaking with a loud metallic click, and the bubble of vapour reappeared: the length of this bubble measured the extension of the liquid. M. Berthelot in this way got extensions of volume of 1/120 for water, 1/93 for alcohol, and 1/59 for ether. Professor Worthington has improved this method by inserting in the liquid an ellipsoidal bulb filled with mercury and provided with a narrow graduated capillary stem; when the liquid is in a state of tension the volume of the bulb expands and the mercury sinks in the stem; from the amount it sinks the tension can be measured. The extension was measured in the same way as in Berthelot's experiments. In this way Professer Worthington showed (*Phil. Trans. A.* 1892, p. 355) that the absolute coefficient of volume elasticity for alcohol is the same for extension as for compression, and is constant between pressures of + 12 and − 17 atmospheres.

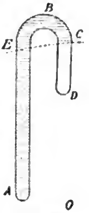

FIG. 85.

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* Dixon and Joly (*Phil. Trans. B.* 1895, p. 568) have shown that air or other gases held in solution do not affect these experiments. The boiling is probably efficacious only in removing bubbles or free gases.

CHAPTER XII.

THE RELATION BETWEEN THE PRESSURE AND VOLUME OF A GAS.

CONTENTS—Boyle's Law—Deviations from Boyle's Law—Regnault's Experiments—Amagat's Experiments—Experiments at Low Pressures—Van der Waals' Equation.

In this chapter we shall confine ourselves to the discussion of the relation between the pressure and the volume of a gas when the temperature is constant and no change of state takes place; the liquefaction of gases will be dealt with in the volume on Heat.

The relation between the pressure and the volume of a given mass of gas was first stated by Boyle in a paper communicated to the Royal Society in 1661. The experiment which led to this law is thus described by him. "We took then a long glass tube, which by a dexterous hand and the help of a lamp was in such a manner crooked at the bottom, that the part turned up was almost parallel to the rest of the tube, and the orifice of this shorter leg of the siphon (if I may so call the whole instrument) being hermetically sealed, the length of it was divided into inches (each of which was subdivided into eight parts) by a straight list of paper, which, containing those divisions, was carefully pasted all along it. Then putting in as much quicksilver as served to fill the arch or bended part of the siphon, that the mercury standing in a level might reach in the one leg to the bottom of the divided paper and just to the same height or horizontal line in the other, we took care, by frequently inclining the tube, so that the air might freely pass from one leg into the other by the sides of the mercury (we took, I say, care), that the air at last included in the shorter cylinder should be of the same laxity with the rest of the air about it. This done, we began to pour quicksilver into the longer leg of the siphon, which by its weight pressing up that in the shorter leg did by degrees strengthen the included air, and continuing this pouring in of quicksilver till the air in the shorter leg was by condensation reduced to take up but half the space it possessed (I say, possessed not filled) before, we cast our eyes upon the longer leg of the glass, on which was likewise pasted a list of paper carefully divided into inches and parts, and we observed not without delight and satisfaction that the quicksilver in that longer part of the tube was 29 inches higher than the other . . . the same air being brought to a degree of density about twice as great as that it had before, obtains a spring twice as strong as formerly." Boyle made a series of measurements with greater compressions until he had reduced the volume to one quarter of its original value, and obtained a close agreement between the pressure observed and "what that pressure should be according to the hypothesis that supposes the pressures and expansions* to be in reciprocal proportions." Although Mariotte did not state the law until fourteen years after Boyle had published his discovery, "the hypothesis that supposes the pressures and expansions to be in reciprocal proportions" is often on the Continent called Mariotte's Law.

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* Or *volumes*, in modern English.

If v is the volume of a given mass of gas and p the pressure to which it is subjected, then Boyle's Law states that when the temperature is constant

$$pv = \text{constant}.$$

Another way of stating this law is that, if ρ is the density of a gas under pressure p,

$$p = \mathrm{R}\rho,$$

where R is a constant when the temperature is constant. Later researches made by Charles and Gay-Lussac have shown how R varies with the temperature and with the nature of the gas. These will be described in the volume on Heat; it will suffice to say here that the pressure of a perfect gas is given by the equation

$$p = \mathrm{KNT},$$

where T is the absolute temperature, N the number of molecules of the gas in unit volume, and K a constant which is the same for all gases.

From the equation $pv = c$ we see that if Δp, Δv are corresponding increments in the pressure and volume of a gas whose temperature is constant, then

$$\Delta p . v + p\Delta v = 0$$

or

$$-\frac{v\Delta p}{\Delta v} = p;$$

but the left-hand side is by definition the bulk modulus of elasticity, hence the bulk modulus of elasticity of a gas at a constant temperature is equal to the pressure.

The work required to diminish the volume of a gas by Δv is $p\Delta v$; the work which has to be done to diminish the volume from v_1 to v_2 is therefore

$$\int_{v_2}^{v_1} p dv,$$

or, since by Boyle's Law $p = c/v$, when the temperature is constant, we see that in this case the work is

$$c\int_{v_2}^{v_1} \frac{1}{v} dv = c \log_e \frac{v_1}{v_2} = p_1 v_1 \log_e \frac{v_1}{v_2},$$

where p_1 is the pressure when the volume is v_1.

Deviations from Boyle's Law.—The first to establish in a satisfactory manner the existence in some gases, at any rate, of a departure from Boyle's Law was Despretz, who, in 1827, enclosed a number of different gases in barometer-tubes of the same length standing in the same cistern. The quantity of the different gases was adjusted so that initially the mercury stood at the same height in the different tubes; pressure was then applied to the mercury in the cistern, so that mercury was forced up the tubes. It was then found that the volumes occupied by the gases were no longer equal, the volumes of carbonic acid and ammonia were less than that of air, while that of hydrogen was greater. This showed that some of the gases did not obey Boyle's Law; it left open the question, however, as to whether any gases did obey it. The next great advance was made by Regnault,[*] who in 1847 settled the question as to the behaviour of certain gases for pressures between 1 and about 30 atmospheres. Regnault's method was to start with a certain quantity of gas occupying a volume v in a tube sealed at the upper end, and with the lower end opening into a closed vessel full of mercury, and then by pumping mercury up a long mercury column rising from the closed vessel to increase the pressure until the volume was halved. By measuring the difference of height of mercury in the column and in the tube the pressure required to do this could be determined. Air under this pressure was now pumped into the closed tube until the volume occupied by the gas was again v; mercury was again pumped up the column until the volume had again been halved and a new reading of the pressure taken; air was pumped in again until the volume was again v, and then the pressure increased again until the volume was halved. In this way the values of pv at a series of different pressures could be compared. The results are shown in the following table; p_0 is given in millimetres of mercury, p_0v_0 is the value of pv at the pressure given in the table, p_1v_1 the value at double this pressure. The experiments were made at temperatures between 2° C. and 10° C.

AIR.		NITROGEN.		CARBONIC ACID.		HYDROGEN.	
p_0	p_0v_0/p_1v_1	p_0	p_0v_0/p_1v_1	p_0	p_0v_0/p_1v_1	p_0	p_0v_0/p_1v_1
738·72	1·001414	753·96	1·001012	764·03	1·007597		
2068·20	1·002709	1159·43	1·001074	1414·77	1·012313		
4219·05	1·003336	2159·22	1·001097	2164·81	1·018973	2211·18	0·998584
6770·15	1·004286	3030·22	1·001950	3186·13	1·028494	3989·47	0·996961
9336·41	1·006366	4953·92	1·002952	4879·77	1·045625	5845·18	0·996121
11472·00	1·005619	5957·96	1·003271	6820·22	1·066137	7074·96	0·994697
		7294·47	1·003770	8393·68	1·084278	9147·61	0·993258
		8628·54	1·004768	9620·06	1·099830	10361·88	0·992327
		9767·42	1·005147				
		10981·42	1·006456				

It will be seen from these figures that between pressures of from about 1 to 30 atmospheres the product pv constantly diminishes for air, nitrogen, and carbonic acid, as the pressure increases, the diminution being most marked for carbonic acid; on the other hand in hydrogen pv increases with the pressure. Natterer, who in 1850 published the results of experiments on the relation between the pressure and volume of a gas at very high pressure, showed that after passing certain pressures pv for air and nitrogen begins to increase, so that pv has a minimum value at a certain pressure; after passing this pressure air and nitrogen resemble hydrogen, and pv continually increases as the pressure increases. This result was confirmed by the researches of Amagat and Cailletet. Each of these physicists worked at the bottom of a mine, and produced their pressures by long columns of mercury in a tube going up the shaft of the mine. Amagat's tube was 300 metres long, Cailletet's 250. Amagat found that the minimum value of pv between 18° and 22° C. occurred at the following pressures:

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[*] *Mémoires de l'Institut de France*, vol. xxi. p. 329.

Nitrogen . .	50 metres of mercury.	Carbon monoxide	50 metres of mercury.
Oxygen . .	100 " "	Marsh gas .	120 " "
Air . . .	65 " "	Ethylene .	65 "

The results of his experiments are exhibited in the following figures; the ordinates are the values of pv, and the abscissæ the pressure, the unit of pressure being the atmosphere, which is the pressure due to a column of mercury 760 mm. high at 0° C., and at the latitude of Paris. The numbers on the curves indicate the temperature at which the experiments were made. It will be noticed that for nitrogen the pressure at which pv is a minimum diminishes as the temperature increases, so much so that at a temperature of about 100° C. the minimum value of pv is hardly noticeable in the curve. This is shown clearly by the following results given by Amagat:

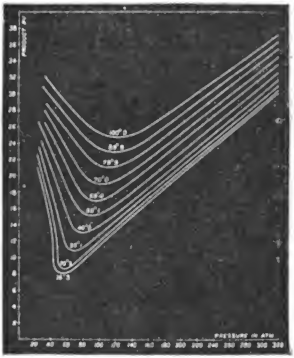

FIG. 86.—Ethylene.

p	17·7° C. pv	30·1° C. pv	50·4° C. pv	75·5° C. pv	100·1° C. pv
30 metres . . .	2745	2875	3080	3330	3575
60 " . . .	2740	2875	3100	3360	3610
100 " . . .	2790	2930	3170	3445	3695
200 " . . .	3075	3220	3465	3750	4020
320 " . . .	3525	3675	3915	4210	4475

Amagat extended his experiments to very much higher pressures, and obtained the results shown in the following table; the temperature was 15° C., and pv was equal to 1 under the pressure of 1 atmosphere:

p (in atmospheres).	Air. pv	Nitrogen. pv	Oxygen. pv	Hydrogen. pv
750	1·650	1·6965	—	—
1000	1·974	2·032	1·735	1·688
1500	2·563	2·644	2·238	2·016
2000	3·132	3·226	2·746	2·322
2500	3·672	3·787	3·235	2·617
3000	4·203	4·338	3·705	2·892

A question of considerable importance in these experiments, and one which we have hardly sufficient information to answer satisfactorily, arises from the condensation of gas on the walls of the manometer, and possibly a penetration of the gas into the substance of these walls. It is well known that when we attempt to exhaust a glass vessel a considerable amount of gas comes off the glass, and if the vessel contains pieces of metal the difficulty of getting a vacuum is still further increased, as gas for some time continues to come from the metal. Much of this is, no doubt, condensed on the surface, but when we remember that water can be forced through gold it seems not improbable that at high pressure the gas may be forced some distance into the metal as well as condensed on its surface.

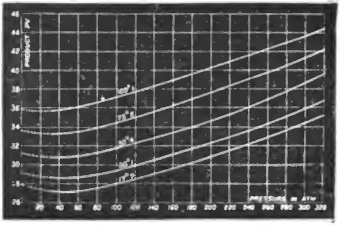

Fig. 87.—Nitrogen.

Boyle's Law at Low Pressures. — The difficulty arising from gas coming off the walls of the manometer becomes specially acute when the pressure is low, as here the deviations from Boyle's Law are so small that any trifling error may completely vitiate the experiments. This is probably one of the reasons why our knowledge of the relation between the pressure and volume of gases at low pressures is so unsatisfactory, and the results of different experiments so contradictory. According to Mendeleeff, and his result has been confirmed by Fuchs, *pv* for air at pressures below an atmosphere diminishes as the pressure diminishes, the value of *pv* changing by about 3·5 per cent. between the pressure of 760 and 14 mm. of mercury. If this is the case, then *pv* for air has a maximum as well as a minimum value. On the other hand, Amagat, who made a series of very careful experiments at low pressures, was not able to detect any departure from Boyle's Law. According to Bohr, and his result has been confirmed by Baly and Ramsay, the law connecting *p* and *v* for oxygen changes at a pressure of about ·75 mm. of mercury. It has been suggested that this is due to the formation of ozone. The recent investigations by Lord Rayleigh on the relation between the pressure and volume of gases at low pressures do not show any departure from Boyle's Law even in the case of oxygen.

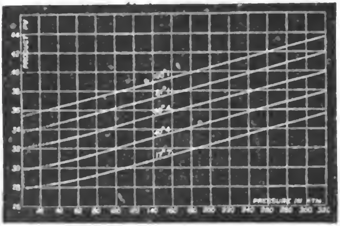

Fig. 88.—Hydrogen.

The results of Amagat's experiments are in fair accordance with the relation between *p* and *v*, arrived at by Van der Waals from the Kinetic Theory of Gases. This relation is expressed by the equation

$$\left(p+\frac{a}{v^2}\right)(v-b)=\mathrm{RT}$$

where a, b, R are constants and T is the absolute temperature. Thus p in Boyle's equation is replaced by $p+a/v^2$ and v by $v-b$. The term a/v^2 or $a\rho^2$, where ρ is the density, arises from the attractions between the molecules of the gas; this attraction assists the outside pressure to diminish the volume of the gas. If we imagine the gas divided by a plane into two portions A and B, then $a\rho^2$ is the attraction of A on B per unit area of the plane of separation; it is the quantity we call the intrinsic pressure in the theory of Capillarity (*see* chap. xv). The v of Boyle's Law is replaced by $v-b$. Since the molecules are supposed to be of a finite although very small size, only a part of the volume "occupied" by the gas is taken up by the molecules, and the actual volume to be diminished is the difference between the space "occupied" by the gas and that filled by its molecules; b is proportional to the volume of a molecule of the gas.

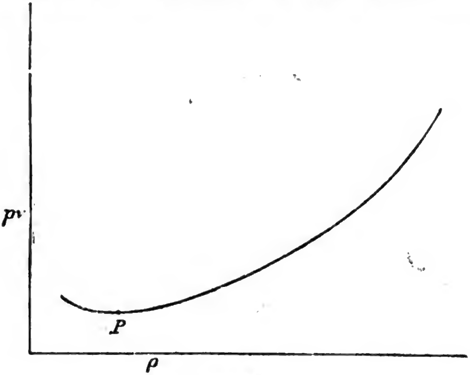

FIG. 89.

Van der Waals' equation may be written:

$$\left(pv+\frac{a}{v}\right)\left(1-\frac{b}{v}\right)=\mathrm{RT}$$

so that if $pv=y$ and $\frac{1}{v}=\rho=x$,

we have $(y+ax)(1-bx)=\mathrm{RT}$

Thus, if the temperature is constant, the curve which represents the relation between pv and ρ is the hyperbola

$$(y+ax)(1-bx)=\text{constant}.$$

The asymptotes of this hyperbola are $y+ax=o$, $1-bx=0$. There is a minimum value of pv at the point P (Fig. 89) where the tangent is horizontal. The value of x at this point is easily shown to be given by the equation

$$a(1-bx)^2 = b\text{RT}.$$

If $b\text{RT}/a$ is less than unity there is a positive value of x given by this equation. This corresponds to the minimum value for pv in the cases of air, nitrogen, and carbonic acid. We see, too, from the equation that as T increases x diminishes, that is, the pressure at which the minimum value of pv occurs is lower at high temperatures than at low. This agrees with the results of Amagat's experiments on nitrogen. When T gets so large that $b\text{RT}/a$ is unity $x=0$; at all higher temperatures it is negative—*i.e.*, P is to the left of the vertical axis, there is thus no minimum value of pv, and the gas behaves like hydrogen in that pv continually increases as the pressure increases.

CHAPTER XIII.

REVERSIBLE THERMAL EFFECTS ACCOMPANYING ALTERATIONS IN STRAINS.

CONTENTS.—Application of Thermodynamics—Ratio of Adiabatic to Isothermal Elasticity.

If the coefficients of elasticity of a substance depend upon the temperature an alteration in the state of strain of a body will be accompanied by a change in its temperature. If the body is stiffer at a high temperature than at a low one, then, if the strain is increased, there will be an increase in the temperature of the strained body, while if the body is stiffer at a low temperature than at a high one, there will be a fall in the temperature when the strain is increased. Thus, if the changes in strain in any experiment take place so rapidly that the heat due to these changes has not time to escape, the coefficients of elasticity determined by these experiments will be larger than the values determined by a method in which the strains are maintained constant for a sufficiently long time for the temperature to become uniform; this follows from the fact that the thermal changes which take place when the strains are variable are always such as to make the body stiffer to resist the change in strain. In those experiments by which the coefficients of elasticity are determined by acoustical methods—*i.e.*, by methods which involve the audible vibration of the substance (*see* Sound, p. 125)—the heat will not have time to diffuse, and we should expect such methods to give higher values than the statical ones we have been describing. When we calculate the ratio of the two coefficients we find that the theoretical difference is far too small to explain the considerable excess of the values of the constants of elasticity found by Wertheim by acoustical methods over those found by statical methods.

We can easily calculate by the aid of Thermodynamics the thermal effects due to a change of strain. To fix our ideas, suppose we have two chambers, one maintained at a temperature T_0, the other at the temperature T_1; these temperatures are supposed to be absolute temperatures, and T_0 to be less than T_1. Let us suppose that we have in the cool chamber a stretched wire, and that we increase the elongation e by δe; then if P is the tension required to keep the wire stretched, the work done on the wire is

$$Pal\delta e$$

where a is the area of the cross-section and l the length of the wire. Now transfer the wire with its length unaltered to the hot chamber, and for simplicity suppose the thermal capacity of the wire exceedingly small, so that we can neglect the amount of heat required to heat up the wire; if the stiffness of the wire changes with temperature the tension P′ required to keep it stretched will not be the same as P. Let the wire contract in the hot chamber until its elongation diminishes by δe, then the work done by the wire is

$$P'al\delta e.$$

Now transfer the wire with its length unaltered back to the cold chamber, it will now be in the same state as when it started. The work done by the wire exceeds that done on it by

$$(P' - P)al\delta e;$$

hence the arrangement constitutes a heat engine, and since it is evidently reversible it must obey the laws of such engines. These engines work by taking heat δH from the hot chamber and giving δh out in the colder chamber, and from the Second Law of Thermodynamics we have

$$\frac{\delta H}{T_1} = \frac{\delta h}{T_0} = \frac{\delta H - \delta h}{T_1 - T_0}$$

Now by the Conservation of Energy

$$\delta H - \delta h = \text{mechanical work done by the engine}$$

$$= (P' - P)\, al\delta e;$$

hence

$$\delta h = T_0 \frac{(P' - P)}{T_1 - T_0} al\delta e$$

$$= T_0 \left(\frac{\delta P}{\delta T}\right)_{e \text{ constant}} al\delta e \; .$$

Now δh is the amount of heat given out by the wire when the elongation is increased by δe, and al is the volume of the wire; hence the mechanical equivalent of the heat given out per unit volume, when the elongation is measured by δe, is equal to

$$T_0 \left(\frac{\delta P}{\delta T}\right)_{e \text{ constant}} \delta e$$

If this heat is prevented from escaping from the wire it will raise the temperature, and if $\delta\theta$ is the rise in temperature due to the elongation δe, we see that

$$\delta\theta = \frac{T_0 \left(\frac{\delta P}{\delta T}\right)_{e \text{ constant}}}{J K \rho} \times \delta e \qquad (1)$$

where ρ is the density of the wire, K its specific heat, and J the mechanical equivalent of heat. We see that this expression proves the statement made above, that the temperature change which takes place on a change in the strain is always such as to make the body stiffen to resist the change.

We can readily obtain another expression for $\delta\theta$, which is often more convenient than that just given. In that formula we have the expression $(\delta P/\delta T)_e$ constant. Now, suppose that, instead of keeping e constant all through, we first allow the body to expand under constant tension; if ω is the coefficient of linear expansion for heat, and δT the change in temperature, the increase in the elongation is $\omega\delta T$; now keep the temperature constant, and diminish the tension until the shortening due to the diminution in tension just compensates for the lengthening due to the rise in temperatures. In order to diminish the elongation by $\omega\delta T$ we must diminish the tension by $q\omega\delta T$ where q is Young's modulus for the wire, hence

$$\delta P = -q\omega\delta T$$

or

$$\left(\frac{\delta P}{\delta T}\right)_{e \text{ constant}} = -q\omega$$

hence by equation (1)

$$\delta\theta = -\frac{T_0 q\omega\delta e}{JK\rho}$$

But $q\delta e$ is the additional tension δP required to produce the elongation δe, hence the increase in temperature $\delta\theta$ produced by an increase of tension δP is given by the equation

$$\delta\theta = -\frac{T_0\omega\delta P}{JK\rho} \qquad (2)$$

Equations (1) and (2) are due to Lord Kelvin.

Dr. Joule (*Phil. Trans.* cxlix. 1859, p. 91) has verified equation (2) by experiments on cylindrical bars of various substances, and the results of his experiments are given in the following table. The changes in temperature were measured by thermo-electric couples inserted in the bars.

	T	ρ	ω	K	δP	$\delta\theta$ observed.	$\delta\theta$ calculated.
Iron . .	286·3	7·5	$1{\cdot}24 \times 10^{-5}$	·110	$1{\cdot}09 \times 10^{9}$	− ·1007	− ·107
Hard steel .	274·7	7·0	$1{\cdot}23 \times 10^{-5}$	·102	$1{\cdot}09 \times 10^{9}$	− ·1620	− ·125
Cast iron .	282·3	6·04	$1{\cdot}11 \times 10^{-5}$	·120	$1{\cdot}10 \times 10^{9}$	− ·1481	− ·115
Copper .	274·2	8·95	$1{\cdot}7182 \times 10^{-5}$	·095	$1{\cdot}08 \times 10^{9}$	− ·174	− ·154

A qualitative experiment can easily be tried with a piece of indiarubber. If an indiarubber band be loaded sufficiently to produce a considerable extension and if it be then warmed by bringing a hot body near to it, it will contract and lift the weight; hence the indiarubber gets stiffer by a rise in temperature; by the rule we have given, it ought to increase in temperature when stretched, since by so doing it becomes stiffer to resist stretching. That this is the case can easily be verified by suddenly stretching a rubber-band and then testing its temperature by placing it against a thermopile, or even between the lips, when it will be found perceptibly warmer than it was before stretching.

We can easily calculate what effect the heat produced will have on the apparent elasticity if it is not allowed to escape. The modulus of elasticity, when the change in strain takes place so rapidly that the heat has not time to escape, is often called the adiabatic modulus.

Ratio of Adiabatic to Isothermal Elasticity.—Suppose we take the case of a wire, and suppose the tension increased by δP, if the heat does not escape the increase δe in the elongation will be due to two causes —one from the increase in the pull, the other from the increase in the temperature. The first part is equal to $\delta P/q$, where q is Young's modulus for steady strain; the second part is equal to $\delta\theta\omega$ where $\delta\theta$ is the change in temperature, ω the coefficient of linear expansion; hence

$$\delta e = \frac{\delta P}{q} + \omega\delta\theta;$$

but by equation (2)

$$\delta\theta = -\frac{\omega T_0 \delta P}{JK\rho};$$

hence

$$\delta e = \frac{\delta P}{q} - \frac{\omega^2 T_0}{JK\rho}\delta P$$

or

$$\frac{\delta e}{\delta P} = \frac{1}{q} - \frac{\omega^2 T_0}{JK\rho}$$

But if q' is the adiabatic "Young's Modulus,"

$$\frac{1}{q'} = \frac{\delta e}{\delta P}$$

$$= \frac{1}{q} - \frac{\omega^2 T_0}{JK\rho} \qquad (3)$$

It follows from this equation that $1/q'$ is always less than $1/q$ or q is always greater than q, as we saw from general reasoning must be the case. By equation (3) we can calculate the value of q'/q. The results for temperature 15°C are given in the following table, taken from Lord Kelvin's article on "Elasticity" in the *Encyclopædia Britannica:*

Substance.	ρ	K	ω	$q/10^{11}$	q'/q deduced from equat. 3.
Zinc	7·008	·0927	·0000249	8·56	1·008
Tin	7·404	·0514	·000022	4·09	1·00362
Silver	10·369	·0557	·000019	7·22	1·00315
Copper	8·933	·0949	·000018	12·20	1·00325
Lead	11·215	·0293	·000029	1·74	1·00310
Glass	2·942	·177	·0000086	6·02	1·000600
Iron	7·553	·1098	·000013	18·24	1·00259
Platinum	21·275	·0314	·0000086	16·7	1·00129

Thus we see that in the case of metals q' is not so much as 1 per cent. greater than q. In Wertheim's experiments, however, the excess of q determined by acoustical methods over q determined by statical methods exceeded in some cases 20 per cent. This discrepancy has never been satisfactorily accounted for.

CHAPTER XIV.

CAPILLARITY.

CONTENTS.—Surface Tension and Surface Energy—Rise of Liquid in a Capillary Tube—Relation between Pressure and Curvature of a Surface—Stability of Cylindrical Film—Attractions and Repulsions due to Surface Tension—Methods of Measuring Surface Tension—Temperature Coefficient of Surface Tension—Cooling of Film on Stretching—Tension of very Thin Films—Vapour Pressure over Curved Surface—Effects of Contamination of a Surface.

THERE are many phenomena which show that liquids behave as if they were enclosed in a stretched membrane. Thus, if we take a piece of bent wire with a flexible silk thread stretching from one side to the other and dip it into a solution of soap and water so as to get the part between the silk and the wire covered with a film of the liquid, the silk thread will be drawn tight as in Fig. 90, just as it would be if the film were tightly stretched and endeavouring to contract so that its area should be as small as possible. Or if we take a framework with two threads and dip it into the soap and water, both the threads will be pulled tight as in Fig. 91, the liquid again behaving as if it were in a state of tension. If we take a ring of wire with a liquid film upon it and then place on the film a closed loop of silk and pierce the film inside the loop, the film outside will pull the silk into a circle as in Fig. 92. The effect is again just the same as it would be if the films were in a state of tension trying to assume as small an area as possible, for with a given circumference the circle is the curve which has the largest area; thus, when the silk is dragged into the circular form, the area of the film outside is as small as possible.

FIG. 90.

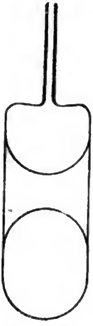
FIG. 91.

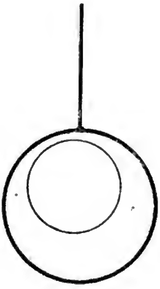
FIG. 92.

Another method of illustrating the tension in the skin of a liquid is to watch the changes in shape of a drop of water forming quietly at the end of a tube before it finally breaks away. The observation is rendered much easier if the water drops are allowed to form in a mixture of paraffin oil and bisulphide of carbon, as the drops are larger and form more gradually. The shape of the drop at one stage is shown in Fig. 93.

If we mount a thin indiarubber membrane on a hoop and suspend it as in Fig. 94, and gradually fill the vessel with water and watch the changes in the shape of the membrane, these will be found to correspond closely to those in the drop of water falling from the tube; the stage corresponding to that immediately preceding the falling away of the drop is especially interesting; a very marked waist forms in the membrane at this stage, and the water in the bag falls rapidly and looks as if it were going to burst away; the membrane, however, reaches another figure of equilibrium, and if no more water is poured in remains as in Fig. 94.

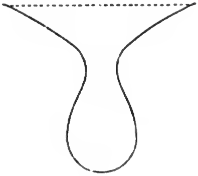

FIG. 93.

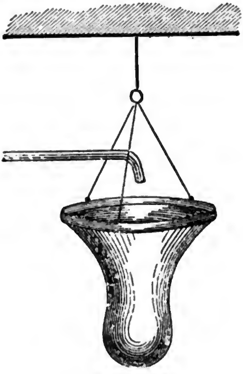

FIG. 94.

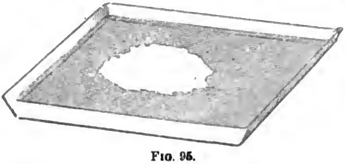

FIG. 95.

Again, liquids behave as if the tension in their outer layers was different for different liquids. This may easily be shown by covering a white flat-bottomed dish with a thin layer of coloured water and then touching a part of its surface with a glass rod which has been dipped in alcohol; the liquid will move from the part touched, leaving the white bottom of the dish dry. This shows that the tension of the water is greater than that of the mixture of alcohol and water, the liquid being dragged away from places where the tension is weak to places where it is strong.

There is one very important difference between the behaviour of ordinary stretched elastic membranes and that of liquid films, for while the tension in a membrane increases with the amount of stretching, the tension in a liquid film is independent of the stretching, provided that this is not so great as to reduce the thickness of the film below about five millionths of a centimetre. This can be shown by the following experiment: bend a piece of wire into a closed plane curve and dip this into a solution of soap and water so as to get it covered with a film, then hold the wire in a nearly vertical position so as to allow the liquid in the film to drain down; this will cause the film to be thinner at the top than at the bottom; the difference in thickness is very apparent when the film gets thin enough to show the colours of thin plates, yet though the film is of very uneven thickness the equilibrium of the film shows that the tension is the same throughout,* for if the tension in the thin part were greater than that in the thick, the top of the film would drag the bottom part up, while if the tension of the thick part were greater than that of the thin the lower part of the film would drag the top part down.

Definition of Surface Tension.—Suppose that we have a film stretched on the framework ABCD, Fig. 96, of which the sides AB, BC and AD are fixed while CD is movable; then, in order to keep CD in equilibrium, a force F must be applied to it at right angles to its length. This force is required to balance the tensions exerted by each face of the film; if T is this tension, then

$$2T.CD = F;$$

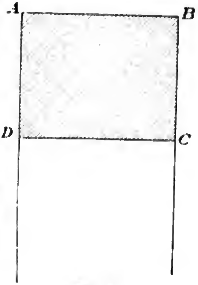

FIG. 96.

the quantity T defined by this equation is called the surface tension of the liquid; for water at 18°C. it is about 73 dynes per centimetre.

Potential Energy of a Liquid arising from Surface Tension.—If we pull the bar CD out through a distance x, the work done is Fx, and this is equal to the increase in the potential energy of the film, but $Fx = 2T.CDx = T \times$ (increase of area of film). Thus the increase in the potential energy of the film is equal to T multiplied by the increase in area, so that in consequence of surface tension a liquid will possess an amount of potential energy equal to the product of the surface tension of the liquid and the area of the surface. Starting from this result we can, as Gauss showed, deduce the consequences of the existence of surface tension from the principle that when a mechanical system is in equilibrium the potential energy is a minimum. Suppose that we take, as Plateau did, two liquids of the same density, say oil and a mixture of alcohol and water, and consider the equilibrium of a mass of oil in the mixture. Since the density of the oil is the same as that of the surrounding fluid, changes in the shape of the mass will not affect the potential energy due to gravity; the only change in the potential energy will be the change in the energy due to surface tension, and, by the principle just stated, the oil will assume the shape in which this potential energy is a minimum—*i.e.*, the shape in which the area of the surface is a minimum. The sphere is the surface which for a given volume has the smallest surface, so that the drops of oil in the liquid will be spherical. This experiment can easily be tried, and the spherical form of the drops is very evident, especially if the oil is made more distinct by the addition of a little iodine.

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* If the film is vertical the tension at the top is very slightly greater than that at the bottom, so as to allow the difference of tension to balance the exceedingly small weight of the film.

If a drop of liquid is not surrounded by fluid of the same density, but is like a drop of mercury on a plate which it does not wet, then any change in the shape of the drop will affect the potential energy due to gravitation as well as that due to surface tension, and the shape of the drop will be determined by the condition that the total potential energy is to be as small as possible; if the drop is very large, the potential energy due to the surface tension is insignificant compared with that due to gravity, and the drop spreads out flat so as to get its centre of gravity low, even though this involves an increase in the potential energy due to the surface-tension. If, however, the drop is very small, the potential energy due to gravity is insignificant in comparison with that due to surface-tension, and the drop takes the shape in which the potential energy due to surface-tension is as small as possible; this shape, as we have seen, is the spherical, and thus surface-tension will cause all very small drops to be spherical. Dew-drops and rain-drops are very conspicuous examples of this; other examples are afforded by the manufacture of spherical pellets by the fall of molten lead from a shot tower and by the spherical form of soap-bubbles. We shall show later on that if the volume of liquid in a drop is the same as that of a sphere of radius a the liquid will remain very nearly spherical if a^2 is small compared with $T/g\rho$ where T is the surface-tension and ρ the density of the liquid. Thus, in the case of water, where T is about 73, drops of less than 2 or 3 millimetres in radius, will be approximately spherical.

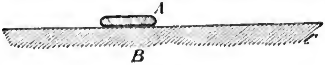

FIG. 97.

Another important problem which we can easily treat by the method of energy is that of the spreading of one liquid over the surface of another. Suppose, for example, we place a drop of liquid A on another liquid B (Fig. 97), we want to know whether A will spread over B like oil over water, or whether A will contract and gather itself up into a drop. The condition that the potential energy is to be as small as possible shows that A will spread over B if doing so involves a diminution in the potential energy; while, if the spreading involves an increase in the potential energy, A will do the reverse of spreading and will gather itself up in a drop. Let us consider the change in the potential energy due to an increase S in the area of contact of A and B where A is a flat drop. We have three surface-tensions to consider: that of the surface of contact between A and the air, which we shall call T_1; that of the surface of contact between B and the air, which we shall call T_2; and that of the surface of contact of A and B, which we shall call T_{12}. Now when we increase the surface of contact between A and B by S we increase the energy due to the surface-tension between these two fluids by $T_{12} \times S$, we increase that due to the surface-tension between A and the air by $T_1 \times S$ and diminish that due to the surface-tension between B and the air by $T_2 \times S$. Hence the total *increase* in the potential energy is

$$(T_1 + T_{12} - T_2)S,$$

and if this is negative S will increase—*i.e.*, A will spread over B; the condition for this to be negative is that

$$T_2 > T_1 + T_{12},$$

so that if this condition is fulfilled the liquid A will spread out into a thin film and cover B, and there will be no place where three liquid surfaces meet. If, on the other hand, any one of the tensions is less than the sum of the other two—*i.e.*, if we can construct a triangle whose sides are proportional to T_1, T_2 and T_{12}, then a drop of one liquid can exist on the surface of the other, and we should have the three liquid surfaces meeting at the edge of a drop. The triangle whose sides are proportional to T_1, T_2, T_{12} is often called Neumann's triangle; the experiments of Quincke, Marangoni and Van Mensbrugghe, show that for all the liquids hitherto investigated this triangle cannot be drawn, as one of the tensions is always greater than the sum of the other two, and hence that there can be no position of equilibrium in which three liquid surfaces meet. Apparent exceptions to this are due to the fouling of the surface of one of the liquids. Thus, when a drop of oil stands on water, the water surface is really covered with a thin coating of oil which has spread over the surface; or again, when a drop of water stands on mercury, the mercury surface is greasy, and the grease has spread over the water. Quincke has shown that a drop of pure water will spread over the surface of pure mercury.

Though three liquid surfaces cannot be in equilibrium when there is a line along which all three meet, yet a solid and two liquid surfaces can be in equilibrium; this is shown by the equilibrium of water or of mercury in glass tubes when we have two fluids, water (or mercury), and air, both in contact with the glass. The consideration of the condition of equilibrium in this case naturally suggests the question as to whether there is anything corresponding to surface-tension at the surface of separation of two substances, one of which is a solid. Though in this case the idea of a skin in a state of tension is not so easily conceivable as for a liquid, yet there is another way of regarding surface-tension which is as readily applicable to a solid as to a liquid. We have seen that the existence of surface-tension implies the possession by each unit area of the liquid of an amount of potential energy numerically equal to the surface-tension: we may from this point of view regard surface-tension as surface energy. There is no difficulty in conceiving that part of the energy of a solid body may be proportional to its surface, and that in this sense the body has a surface-tension, this tension being measured by the energy per unit area of the surface.

Let us now consider the equilibrium of a liquid in contact with air and both resting on a solid, and not acted upon by any forces except those due to surface-tension. Suppose A, Fig. 98, represents the solid, B the liquid, C the air, FG the surface of separation of liquid and air, ED the surface of the solid. Let the angle FGD be denoted by θ; this angle is called the angle of contact of the liquid with the solid. Let the surface of separation FG come into the position F′G′ parallel to FG. Then if FG represented a position of equilibrium, the potential energy due to surface-tension must be a minimum in this position, so that it will be unaffected by any small displacement of the substances; thus the potential energy must not be altered by the displacement of FG to F′G′. This displacement of the surface causes B to cover up a long strip of the solid, the breadth of the strip being GG′. Let S be the area of this strip. Then if T_1, T_2 and T_{12} are respectively the surface-tensions between A and C, B and C, and A and B, the changes in the energy due to the displacement are:

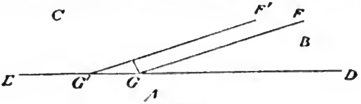

Fig. 93.

(1) An increase $T_{12}S$ due to the increase S in the surface between A and B.

(2) An increase $T_2S\cos\theta$ due to the increase $S\cos\theta$ in the surface between B and C.

(3) A diminution T_1S due to the diminution S in the surface between A and C.

Hence the total increase in the energy is

$$S(T_{12} + T_2\cos\theta - T_1)$$

and as this must vanish when we have equilibrium we have

$$T_{12} + T_2\cos\theta = T_1;$$

or
$$\cos\theta = \frac{T_1 - T_{12}}{T_2}$$

Thus, if T_1 is greater than T_{12}, $\cos\theta$ is positive and θ is less than a right angle; if T_1 is less than T_{12}, $\cos\theta$ is negative, and θ is greater than a right angle; mercury is a case of this kind, as for this substance θ is about 140°. The angle θ is termed the angle of contact. Since $\cos\theta$ cannot exceed unity, the greater of the two quantities T_1 or T_{12} must be less than the sum of the other two. If this condition is not fulfilled the liquid B will spread over the surface A.

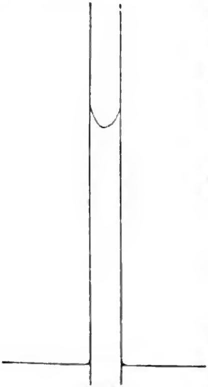

Fig 94.

Rise of a Liquid in a Capillary Tube.—We can apply the result we have just obtained to find the elevation or depression of a liquid in a tube which it does not wet and with which it has a finite angle of contact.

Suppose h is the height of the fluid in the tube above the horizontal surface of the fluid outside, when there is equilibrium; and suppose that r is the radius of the tube at the top of the fluid column. Let T_1 be the surface-tension between the tube and air, T_2 that between the liquid and air and T_{12} that between the tube and the liquid. Then, if there is equilibrium, a slight displacement of the fluid up the tube will not alter the potential energy. Suppose then that the fluid rises a short distance x in the tube, thus covering an additional area $2\pi rx$ of the tube, and diminishing the area of the tube in contact with the air by this amount. This increases the potential energy due to surface-tension by $2\pi rx(T_{12} - T_1)$.

The increase in the potential energy due to gravity is the work done (1) by lifting the mass $\pi r^2 \times \rho \times x$, where ρ is the density of the liquid, against gravity through a height h—this is equal to $g\rho h\pi r^2 x$; and (2) by lifting the volume v of the meniscus through a height x—this work is equal to $g\rho vx$.

Hence the total increase in potential energy is

$$2\pi rx(T_{12} - T_1) + g\rho h\pi r^2 x + g\rho vx,$$

and as this must vanish we have

$$h + \frac{v}{\pi r^2} = \frac{2(T_1 - T_{12})}{g\rho r},$$

but if θ is the angle of contact, we have just proved that

$$T_2 \cos\theta = T_1 - T_{12}$$

hence

$$h + \frac{v}{\pi r^2} = \frac{2T_2 \cos\theta}{g\rho r}$$

When the fluid wets the tube θ is zero and $\cos\theta = 1$. If the meniscus is so small that it may be regarded as bounded by a hemisphere, v is the difference between the volume of a hemisphere and that of the circumscribing cylinder—*i.e.*,

$$v = \pi r^3 - \frac{2}{3}\pi r^3 = \frac{\pi r^3}{3}$$

hence

$$h + \frac{1}{3}r = \frac{2T_2}{g\rho r}$$

If θ is greater than a right angle h is negative, that is, the level of the liquid in the tube is lower than the horizontal surface; this is strikingly shown by mercury, but by no other fluid. The angle of contact between mercury and glass was measured by Gay Lussac by causing mercury to flow up into a spherical glass bulb; when the mercury is in the lower part of the bûlb the surface near the glass will be very much curved; as the mercury rises higher in the bulb the curvature will get less; the surface of the mercury at different levels is represented by the dotted lines in Fig. 100. There is a certain level at which the surface will be horizontal; at this place the tangent plane to the sphere makes with a horizontal plane an angle equal to the supplement of the angle of contact between mercury and glass. A modification of this method is to make a piece of clean plate glass dipping into mercury rotate about a horizontal axis until the surface of the mercury on one side of the plate is flat; the angle made by the glass plate with the horizontal is then the supplement of the angle of contact between mercury and glass.

The angle of contact between mercury and glass varies very widely under different circumstances; thus the meniscus of the mercury in a thermometer may not be the same when the mercury is rising as when it is falling. We should expect this to be the case if the mercury fouls the glass, for in this case the mercury when it falls is no longer in contact with clean glass but with glass fouled by mercury, and we should expect the angle of contact to be very different from that with pure glass. Quincke found that the angle of contact of a drop of mercury on a glass plate steadily diminished with the time; thus the angle of contact of a freshly formed drop was 148° 55′, and this steadily diminished, and after two days was only 137° 14′; on tapping the plate the angle rose to 141° 19′, and after another two days fell to 140°.

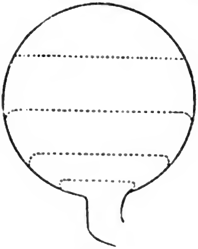

Fig. 100.

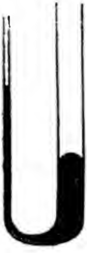

Fig. 101.

If we force mercury up a narrow capillary tube and then gradually diminish the pressure, the mercury at first, instead of falling in the tube, adjusts itself to the diminished pressure by altering the curvature of its meniscus, and it is only when the fall of pressure becomes too large for such an adjustment to be possible that the mercury falls in the tube; the consequence is that the fall of the mercury, instead of being continuous, takes place by a series of jumps. This effect is illustrated by the old experiment of bending a piece of capillary tubing into a U-tube (Fig. 101), pouring mercury into the tube until it covers the bend and stands at some height in either leg of the tube; if the tube is vertical, the mercury can be made by tapping to stand at the same height in both legs of the tube. Now slowly tilt the tube so as to cause the mercury to run up the left leg of the tube; if the tube is slowly brought back to the vertical, the mercury will be found to stand at a higher level in the left leg of the tube than in the right, while the meniscus will be flatter on the left than on the right. This principle explains the action of what are called Jamin's tubes, which are simply capillary tubes containing a large number of detached drops of liquid; these can stand an enormous difference of pressure between the ends of the tube without any appreciable movement of the drops along the tube. Thus, suppose that AB, CD, EF (Fig. 102) represent three consecutive drops along the tube, then in consequence of the different curvatures

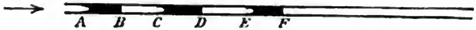

Fig. 102.

of AB at A and B the pressure in the air at A will be greater than that at B, while the pressure at C will be greater than that at D, and so on; thus each drop transmits a smaller pressure than it receives; if we have a large number of drops in the tube the difference of pressure at the ends arising in this way may amount to several atmospheres.

Relation between Pressure and Curvature of a Surface.—If we have a curved liquid surface in a state of tension the pressure on the concave side of the surface must be greater than that on the convex; we shall proceed to find the relation between the difference of pressure on the two sides and the curvature of the surface.

Let the small portion of a liquid film, represented in Fig. 103 by ABCD where AB and CD are equal and parallel and at right angles to AD and BC, be in equilibrium under the surface tension and a difference of pressure p between the two sides of the film. When a system of forces acting on a body are in equilibrium we know by Mechanics that the algebraical sum of the work done by these forces when the body suffers a small displacement is zero. Let the film ABCD (Fig. 103) be displaced so that each point of the film moves outward along the normal to its surface through a small distance x, and let A′B′C′D′ be the displaced position of ABCD. Then the work done by the pressure is equal to

$$p \times \text{area ABCD} \times x;$$

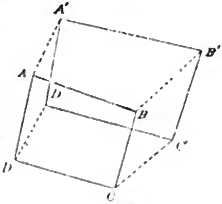

Fig. 103.

the work done *against* the surface tension is T × increase in area of the surface; and since a film has two sides the increase in the area of the film is twice the difference between the areas A′B′C′D′ and the area ABCD. Hence the work done against surface tension is equal to

$$2\text{T} \times (\text{area A}'\text{B}'\text{C}'\text{D}' - \text{area ABCD})$$

Hence by the mechanical principle referred to

$$p \times \text{area ABCD} \times x = 2\text{T}(\text{area A}'\text{B}'\text{C}'\text{D}' - \text{area ABCD}) \qquad (1)$$

if we are considering a drop of water instead of a film we must write T instead of 2T in this equation.

Spherical Soap-bubble.—In this case ABCD will be a portion of a spherical surface and the normals AA′, BB′, CC′, DD′ will all pass through O, the centre of the sphere. Let R be the radius of the sphere, then by similar triangles

$$\text{A}'\text{B}' = \text{AB}\frac{\text{OA}'}{\text{OA}} = \text{AB}\left(1 + \frac{x}{\text{R}}\right)$$

$$\text{B}'\text{C}' = \text{BC}\frac{\text{OB}'}{\text{OB}} = \text{BC}\left(1 + \frac{x}{\text{R}}\right)$$

The area $$\mathrm{A'B'C'D'} = \mathrm{A'B'}.\,\mathrm{B'C'} = \mathrm{AB}.\,\mathrm{BC}\left(1+\frac{x}{\mathrm{R}}\right)^2$$

$$= \mathrm{AB}.\,\mathrm{BC}\left(1+\frac{2x}{\mathrm{R}}\right)$$

as we suppose x/R is so small that its square can be neglected.

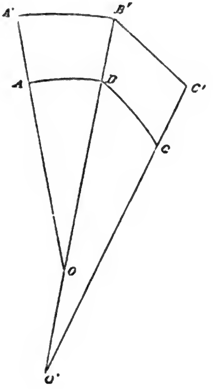

Fig. 104.

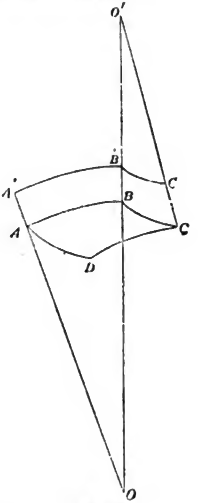

Fig. 105.

Hence $$\text{area } \mathrm{A'B'C'D'} = \text{area } \mathrm{ABCD}\left(1+\frac{2x}{\mathrm{R}}\right) \qquad (2)$$

substituting this value for the area A'B'C'D' in equation (1), the equation becomes

$$p = \frac{4\mathrm{T}}{\mathrm{R}}$$

so that the pressure inside a spherical soap-bubble exceeds the pressure outside by an amount which is inversely proportional to the radius of the bubble.

General Case of a Curved Soap-bubble.—If the element of the film ABCD forms a portion of a curved surface, we know from the theory of such surfaces that we can find two lines AB, BC at right angles to each other on the surface such that the normals to the surface at A and B intersect in O, while those at B and C intersect in a point O′. The lines AB, BC are said to be elements of the curves of Principal Curvature of the surface, and AO and BO′ are called the Radii of principal curvature of the surface. We must now distinguish between two classes of surfaces. In the first class, which includes spheres and ellipsoids, the two points O and O′ are on the same side of the surface, and the surfaces are called synclastic surfaces; in the second class, which includes surfaces shaped like a saddle or a dice-box, O and O′ are on opposite sides of the surface; and the surfaces are called anti-clastic surfaces. We shall consider these cases separately, and take first the case of synclastic surfaces. In this case (Fig. 104) we have by similar triangles

$$A'B' = AB\frac{OA'}{OA} = AB\left(1+\frac{x}{R}\right)$$ if R is the radius of principal curvature OA.

Similarly $B'C' = BC\left(1+\frac{x}{R'}\right)$ if R′ is the radius of principal curvature O′B.

Hence area $A'B'C'D' = \text{area } ABCD\left(1+\frac{x}{R}\right)\left(1+\frac{x}{R'}\right)$

$$= \text{area } ABCD\left(1+x\left(\frac{1}{R}+\frac{1}{R'}\right)\right)$$

as we suppose x/R, x/R' both so small that we can neglect the product of these quantities in comparison with their first powers. Substituting this value for the area A′B′C′D′ in equation (1) we get

$$p = 2T\left(\frac{1}{R}+\frac{1}{R'}\right) \qquad (3)$$

Let us now take the case of an anti-clastic surface, represented in Fig. 105. In this case we have

$$A'B' = AB\left(1+\frac{x}{R}\right)$$

$$B'C' = BC\frac{O'B'}{OB} = BC\left(1-\frac{x}{R'}\right)$$

hence area $A'B'C'D' = \text{area } ABCD\left(1+x\left(\frac{1}{R}-\frac{1}{R'}\right)\right)$

Substituting this value of the area A′B′C′D′ in equation (1) we get

$$p = 2T\left(\frac{1}{R}-\frac{1}{R'}\right) \qquad (4)$$

We can include (3) and (4) in the general formula

$$p = 2T\left(\frac{1}{R}+\frac{1}{R'}\right)$$

if we make the convention that the radius of curvature is to be taken as positive or negative according as the corresponding centre of curvature is on the side of the surface where the pressure is greatest or on the opposite side.

When a soap film is exposed to equal pressures on the two sides $p=0$, and we must therefore have

$$\frac{1}{R}+\frac{1}{R'}=0,$$

In this case the curvature in any normal section must be equal and opposite to the curvature in the normal section at right angles to the first. By stretching a film on a closed piece of wire and then bending the wire we can get an infinite number of surfaces, all of which possess this property; we can also get surfaces with this property by forming a film between the rims of two funnels open at the end, as in Fig. 106. By moving the funnels relatively to each other we get a most interesting series of surfaces, all of which have their principal curvatures equal and opposite.

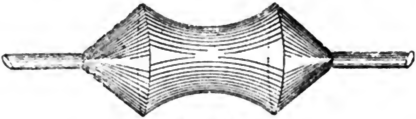

FIG. 106.

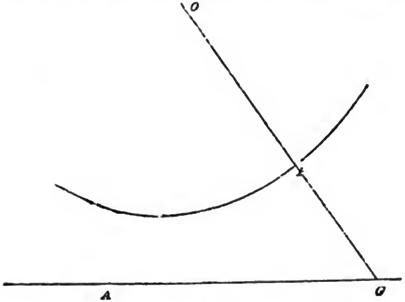

FIG. 107.

If the film is in the shape of a surface of revolution—*i.e.*, one which can be traced out by making a plane curve rotate about a line in its plane—we know from the geometry of such surfaces that (Fig. 107)

$$R=PO \qquad R'=PG$$

where O is the centre of curvature of the plane curve at P, and G the point where the normal at P cuts the axis AG about which the curve rotates.

if the pressures on the two sides of the film are equal we must have PO = − PG.

The only curve with this property is the catenary, the curve in which a uniform heavy string hangs under gravity, and this, therefore, is the shape of the cross-section of a soap film forming a surface symmetrical about an axis, when the pressures on the two sides are equal.

Stability of Cylindrical Films.—Let us consider the case of a symmetrical film whose surface approaches closely that of a right circular cylinder. Let EPF be the curve which by its rotation about the straight line AB generates the surface occupied by the film. EPF will not differ much from a straight line, and PG, the normal at P, will be very nearly equal to PN where PN is at right angles to AB. Hence, if R is the radius of curvature at P and p the constant difference of pressure between the inside and outside of the film, we have

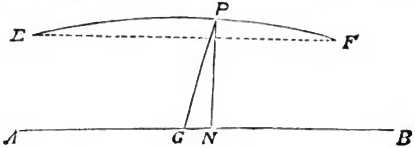

Fig. 108.

$$p = 2T\left(\frac{1}{R} + \frac{1}{PN}\right) \qquad (1)$$

Let y be the height of P above the straight line EF and a the distance between the lines EF and AB, then

$$PN = a + y$$

and as y is very small compared with a we have approximately

$$\frac{1}{PN} = \frac{1}{a} - \frac{y}{a^2}$$

Substituting this value of 1/PN in equation (1) we get

$$\frac{1}{R} = \frac{p}{2T} - \frac{1}{a} + \frac{y}{a^2} = \frac{1}{a^2}\left\{y + a^2\left(\frac{p}{2T} - \frac{1}{a}\right)\right\} = \frac{y'}{a^2} \qquad (2)$$

if y' is the distance of P from a horizontal line at a distance

$$a^2\left(\frac{p}{2T} - \frac{1}{a}\right)$$

below EF. Since the film is very nearly cylindrical, p is very nearly equal to $2T/a$, so that the distance between this line and EF will be very small.

Hence we see from equation (2) that the reciprocal of the radius of curvature at a point on the curve is proportional to the distance of the point from a straight line. Now we saw (p. 96) that the path described by a point fixed near to the centre of a circle when the circle rolls on a straight line possesses this property, hence we conclude that the cross-section of a nearly cylindrical film is a curve of this kind. The curve possesses the following properties: it cuts the straight line, which is the path of the centre of the circle, in a series of points separated by half the circumference of the rolling circle, its greatest distance from this line

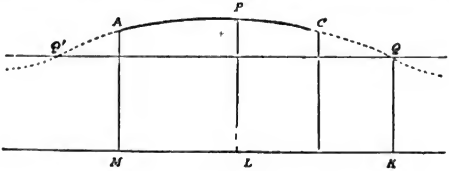

FIG. 109.

is equal to the distance of the point from the centre of the rolling circle, while the reciprocal of the radius of curvature at a point is proportional to its distance from this line.

Let us now consider what is the pressure in a nearly cylindrical bubble with a slight bulge. Let us suppose that the length of the bubble is less than the distance between two points where the curve which generates the surface crosses the path of the centre of the rolling circle. The section of the bubble must form a part of this curve. Let A and C, Fig. 109, be the ends of the bubble APC, the section of the film. Let the dotted line denote the completion of the curve of which APC forms a part. Then if p is the excess of pressure

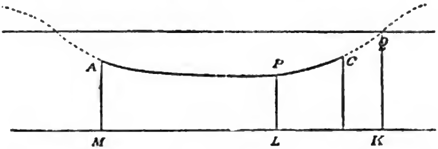

FIG. 110.

inside the bubble over the outside pressure and P any point on the curve,

$$p = 2T\left(\frac{1}{\rho} + \frac{1}{PL}\right)$$

where ρ is the radius of curvature of the curve at P. Now if we take P at Q, a point where the curve crosses its axis $1/\rho = 0$, hence

$$p = \frac{2T}{QK}$$

Now if the film were straight between A and C the excess of pressure p' would be given by the equation

$$p' = \frac{2T}{AM}.$$

As QK is less than AM, p is greater than p', hence the pressure in the film which bulges out is greater than the pressure in the straight film. We can prove in the same way that in a film that bends in, as in Fig. 110, if the distance between the ends is less than the distance between the points Q and Q′ on the curve, that is, if the length of the film is less than half the circumference of its ends, the pressure is less than the pressure in the straight film.

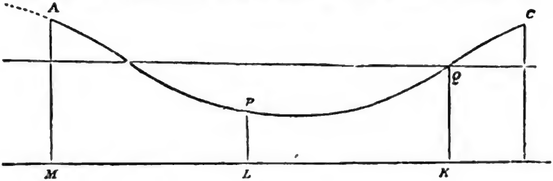

FIG. 111.

If the distance between the ends of the film is greater than half the circumference of the ends of the film these conditions are reversed.

For let Fig. 111 represent such a film bending in; as before, the excess of pressure p will be given by the equation

$$p = \frac{2T}{QK}$$

where Q is the point where the curve of the film crosses its axis. If the film were straight between A and C, p', the excess of pressure, would be given by the equation

$$p' = \frac{2T}{AM}$$

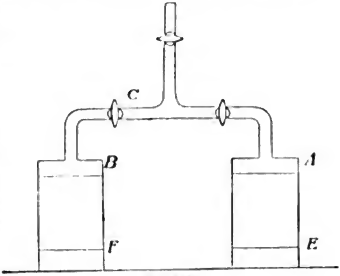

FIG. 112.

Since in this case AM is greater than QK, p' is less than p. Hence the pressure in the film which bends in is greater than that in the straight film. In a similar way we can prove that in this case the pressure in a film which bulges out is less than the pressure in a straight film. Hence we arrive at the result that, if the length of the film is less than half the circumference of its end, the pressure in a film that bulges out is greater than that in a film which bends in, while if the length of the film is greater than its semi-circumference the pressure in the film that bulges out is less than the pressure in one that bends in. Mr. Boys has devised a very beautiful experiment which illustrates this point. The arrangement is represented in Fig. 112. A and B are pieces of glass tubing of equal diameter communicating with each other through the tube C; this communication can be opened or closed by turning the tap. E and F are pieces of glass tubing of the same diameter as A; they are placed vertically below A and B respectively. The distance between A and E and B and F can be altered by raising or lowering the system ABC. First begin with this distance less than half the circumference of the glass tube, Fig. 113, close the tap and blow between A and E a bubble which bulges out, and between B and F, one that bends in. Now open the tap; they will both tend to straighten, air going from the one at A to help to fill up that at B, showing that the pressure in the one at A is greater than in that at B. Now repeat the experiment after increasing the distance between A and E and B and F to more than half the circumference of the tube. We now find on opening the tap that the film which bulges out is blown out still more, while the one that bends in tends to shut itself up, showing that air has gone from B to A or that now the pressure at B is greater than that at A.

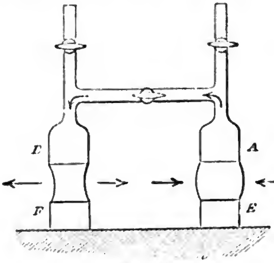

Fig. 113.

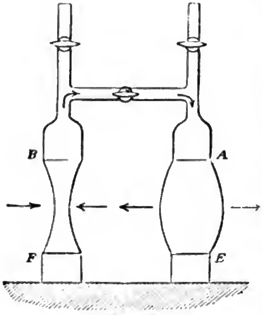

Fig. 114.

It follows from this result that the equilibrium of a cylindrical film is unstable when its length is greater than its circumference, while shorter films are stable.

For let us consider the equilibrium of a cylindrical film between two equal fixed discs, A and B, Fig. 115, and consider the behaviour of a movable disc C of the same size placed between them. Suppose the length of the film is less than its circumference and that C is midway between A and B; move C slightly towards B, then the film between B and C will bulge out while that between A and C will bend in. As the distance between each of the films is less than half the circumference the pressure in the film which bulges out will be greater than in that which bends in, thus C will be pushed back to its original position and the equilibrium will be stable. If C is not midway between AB but nearer to B than to A, then even if AC is greater than the semi-circumference so that when C is pushed towards B the pressure in AC is greater than when the film is straight, yet it is easy to prove that the excess of pressure in BC is, in consequence of its greater curvature, greater than that in AC, so that C is again pushed back to its old position and the film is again stable.

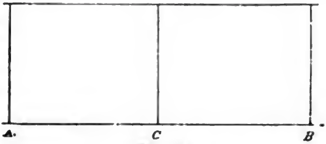

Fig. 115.

Suppose now that the distance between A and B is greater than the circumference of the film, and that C, originally midway between A and B, is slightly displaced towards B. CB will bulge out and CA will bend in; as the length of each of these films is greater than the semi-circumference of the film the pressure in BC will be less than that in AC, and C will be pushed still further from its original position and the equilibrium will be unstable. The film will contract at one part and expand in another until its two sides come into contact and the film breaks up into two separate spherical portions.

These results apply to fluid cylinders as well as to cylindrical films. Such cylinders are unstable when their length is greater than their circumference. Examples of this unstability are afforded by the breaking up of a liquid jet into drops. The development of inequalities in the thickness of the jet is shown in Figs. 116 and 117 taken from instantaneous photographs. The little drops between the big ones are made from the narrow necks which form before the jet finally breaks up. Another instance of this instability is afforded by dipping a glass fibre in water, the water gathers itself up into beads. A very beautiful illustration of the same effect is that of a wet spider's web, shown in Fig. 118, when again the water gathers itself up into spherical beads.

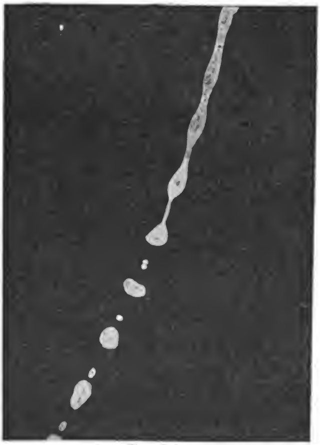

Fig. 116.

If the fluid is very viscous the effect of viscosity may counterbalance the instability due to surface tension; thus it is possible to get long thin threads of treacle or of molten glass and quartz.

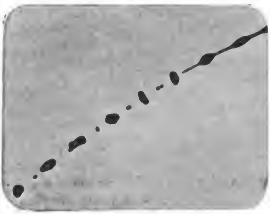

FIG. 117.

Force between two Plates due to Surface-tension.—Let A and B (Fig. 119) be two parallel plates separated by a film of water or some liquid which wets them; then, if d is the distance between the plates and D the diameter of the area of the plate wet by the liquid, the radii of curvature at the free surface of the liquid are approximately $-d/2$ and $D/2$, hence the pressure inside the film is less than the atmospheric pressure by

$$2T\left\{\frac{1}{d}-\frac{1}{D}\right\}$$

or if d is very small compared with D the difference of pressure is approximately $\frac{2T}{d}$.

Now the plate A is pressed towards B by the atmospheric pressure and away from B by a pressure which is less than this by $2T/d$; hence, if A is the area of the plate wet by the film, the force urging A towards B is $\frac{2AT}{d}$.

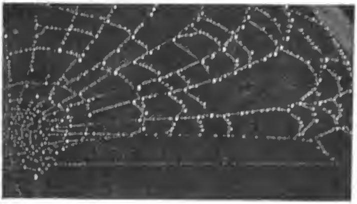

FIG. 118.

The force varies inversely as the distance between the plates; thus, if a drop of water is placed between two plates of glass the plates are forced together, and this still further increases the pull between the plates as the area of the wetted surface increases while the distance between the plates diminishes.

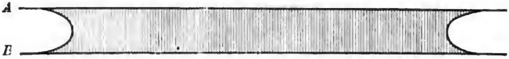

FIG. 119.

Attractions and Repulsions of small Floating Bodies.—Small bodies, such as straw or pieces of cork, floating on the surface of a liquid often attract each other and collect together in clusters; this occurs when the bodies are all wet by the liquid, and also when none of them are wet; if one body is wet and one is not wet they repel each other when they come close together. To investigate the theory of this effect, let us suppose that A and B are two parallel vertical plates immersed in a liquid which wets both of them, the liquid will stand at a higher level between the plates than it does outside. We shall begin by showing that the horizontal force exerted on a plate by a meniscus such as PRQ, UVW is the same as the force which would be exerted if the meniscus were done away with and the liquid continued horizontally up to the surface of the plate. For consider the water in the meniscus PQR; it is in equilibrium under the horizontal tension at P, the vertical tension at Q, the force exerted by the plate on the liquid, the vertical liquid pressure over PR, and the pressure of the atmosphere over PQ. The resultant pressure of the atmosphere over PQ, which we shall call π, in the horizontal direction is equal to the pressure which would be exerted on QR, the part of the plate wet by the meniscus, if this were exposed directly to the atmospheric pressure without the intervention of the liquid. The horizontal forces acting from left to right on the meniscus are

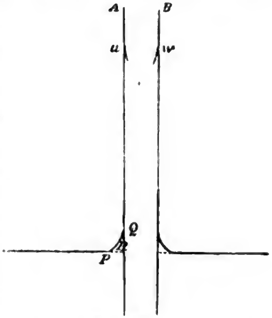

Fig. 120.

$$\pi - \mathrm{T} - \text{force exerted by plate on meniscus.}$$

Since the meniscus is in equilibrium the horizontal forces must be in equilibrium; hence

$$\text{force exerted by meniscus on plate} = \mathrm{T} - \pi,$$

but this is precisely the force which would be exerted if the meniscus were done away with and the horizontal surface of the liquid prolonged to meet the plate. Hence, as far as the horizontal forces are concerned, we may suppose the surfaces of the liquid flat, and represented by the dotted lines in Fig. 120. Considering now the forces acting on the plate A, the pulls exerted by the surface-tension at R and U are equal and opposite; on the left the plate is acted on by the atmospheric pressure, on the right by the pressure in the liquid. Now the pressure in the liquid at any point is less than the atmospheric pressure by an amount proportional to the height of the point above the level of the undisturbed liquid; thus the pressure on A tending to push it towards B is greater than the pressure tending to push it away from B, and thus the plates are pulled together.

Now suppose neither of the plates is wet by the liquid—a case represented in Fig. 121. We can prove, as before, that we may suppose the fluid to be prolonged horizontally to meet the plates. The force tending to push the plate A towards B is the pressure in the liquid, the force

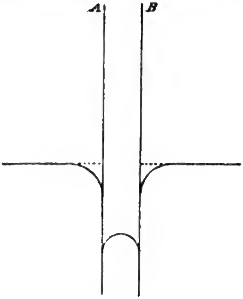

Fig. 121.

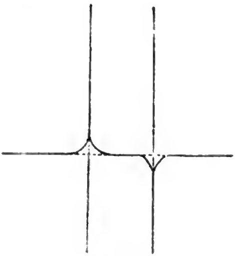

Fig. 122.

tending to push it away is the atmospheric pressure. Now the pressure at any point in the liquid is greater than the atmospheric pressure by an amount proportional to the depth of the point below the undisturbed surface of the liquid; hence, the pressure tending to push A to B will be greater than that tending to push it away from B, so that the plates will again appear to attract each other.

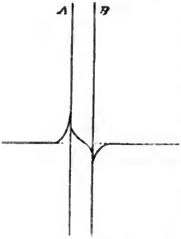

Fig. 123.

Now take the case where one plate is wet by the liquid while the other is not. The section of the liquid surface will be as in Fig. 122, the curvature of the surface being of one sign against one plate, and of the opposite sign against the other. When the plates are a considerable distance apart, the surfaces of the liquid will be like that shown in Fig. 122; between the plates there is a flat horizontal surface at the same level as the undisturbed liquid outside the plates; in this case there is evidently neither attraction nor repulsion between the plates. Now suppose the plates pushed nearer together, this flat surface will diminish, and the last trace of it will be a horizontal tangent crossing the liquid. Since the curvature changes sign in passing from A to B, there must be a place between A and B where it vanishes, and when the curvature vanishes, the pressure in the liquid is equal to the atmospheric pressure; this point, at which the tangent crosses the surface, must be on the prolongation of the free surface of the liquid. Now suppose that the plates are so near together that this tangent ceases to be horizontal, and the liquid takes the shape shown in Fig. 123. We can show, by the method given on p. 153, that the action on the plate A of the meniscus inside A is the same as if the meniscus were removed and the liquid surface stretched horizontally between the plates, the surface-tension in this surface being equal to the *horizontal component* of the surface tension at the point of inflection. Now consider the plate A; it is pulled from B by the surface-tension and towards it by only the horizontal component of this. The force pulling it away is thus greater than the other, and the plates will therefore repel each other. If the plates are pushed very near together so that the point of inflection on the surface gets suppressed the liquid may rise between the plates and the repulsion be replaced by an attraction.

Methods of Measuring Surface-tension.

By the Ascent of the Liquid in a Capillary Tube.—A finely divided glass scale is placed in a vertical position by means of a plumb line, the lower end of the scale dipping into a vessel V, which contains some of the liquid whose surface tension is to be determined. The capillary tube is prepared by drawing out a piece of carefully cleaned glass tube until the internal diameter is considerably less than a millimetre; the bore of the tube should be as uniform as possible, for although the height to which the fluid rises in the capillary tube depends only on the radius of the tube at the top of the meniscus, yet when we cut the tube at this point to determine its radius, if the tube is of uniform bore, no error will ensue if we fail to cut it at exactly the right place. Attach the capillary tube to the scale by two elastic bands, and have a good light behind the scale. Dip the capillary tube in the liquid, and it will rush up the tube; then raise the capillary tube, keeping its end below the fluid in V. This will make the meniscus sink in the tube and ensure that the tube above the meniscus is wetted by the liquid. Now read off on the scale the levels of the liquid in V and the capillary tube, and the difference of levels will give the height to which the liquid rises in the tube. To measure r, the radius of the tube at the level of the meniscus, cut the capillary tube carefully across at this point and then measure the internal radius by a good microscope with a micrometer scale in the eyepiece. If the section, when observed in the microscope, is found to be far from circular, the experiment should be

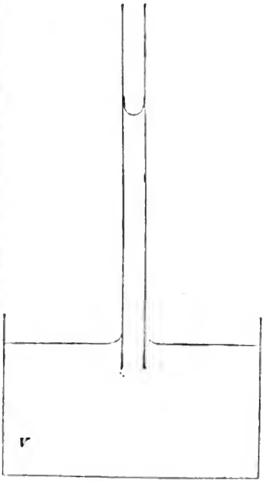

FIG. 121.

repeated with another tube. The surface tension T is determined by the equation (p. 141).

$$T = \frac{1}{2}\rho g\left(hr + \frac{r^2}{3}\right) \text{ where } \rho \text{ is the density of the fluid.}$$

If the angle of contact is not zero a knowledge of its value is required before T can be determined by this method.

By the Measurements of Bubbles and Drops.—This method is due to Quincke. The theory is as follows: suppose that AB, Figs. 125 and 126, represents the section of a large drop of mercury on a horizontal glass plate or, when turned upside down a large bubble of air under a glass plate in water. Let a central slab be cut out of the drop or bubble by two parallel vertical planes unit distance apart, and suppose that this slab is cut in half by a vertical plane at right angles to its length; consider the equilibrium of the portion of this slab above the horizontal section BC of greatest area in the case of the drop, and below it in the case of the bubble.

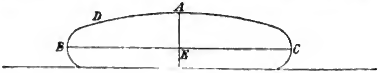

FIG. 125.

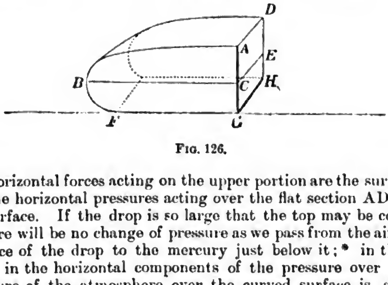

FIG. 126.

The horizontal forces acting on the upper portion are the surface tension T, and the horizontal pressures acting over the flat section ADEC and the curved surface. If the drop is so large that the top may be considered as plane there will be no change of pressure as we pass from the air just above the surface of the drop to the mercury just below it;* in this case the difference in the horizontal components of the pressure over ADEC and the pressure of the atmosphere over the curved surface is, since AD is unity, equal to

$$\tfrac{1}{2}g\rho DE^2$$

As this must be balanced by the surface tension over AD we must have

$$T = \tfrac{1}{2}g\rho DE^2 \qquad (1)$$

By considering the equilibrium of the portion ABFGHD of the drop we have

$$T(1 + \cos\omega) = \tfrac{1}{2}g\rho h^2 \qquad (2)$$

where h is the thickness of the bubble or drop, and ω the angle of contact at F between the liquid and the plate. From equation (2) we have

---

* If the drops are not large enough for this assumption to be true, a correction has to be applied to allow for the difference in pressure on the two sides of the surface through A.

$$h^2 = \frac{4T\cos^2\frac{\omega}{2}}{g\rho}$$

Thus the thickness of all large drops or bubbles in a liquid is independent of the size of the drops or bubbles. By measuring either DE or h, and using equation (1) or (2) we can determine T. In the case of bubbles it is more convenient to use, instead of a flat piece of glass, the concave surface of a large lens, as this facilitates greatly the manipulation of the bubble. In this case, if we use equation (2), we must remember that h is the depth of the bottom of the bubble below the horizontal plane through the circle of contact of the liquid with the glass. Thus, in Fig. 127, h is equal to NE and not to AE. It is more convenient to measure AE and then to calculate NE from the radius of curvature of the lens and the radius of the circle of contact of the glass and the liquid. Determinations of the surface tension of liquids by this method have been made by Quincke, Magie, and Wilberforce.* Magie used this method to determine the angle of contact, as it is evident from equations (1) and (2) that

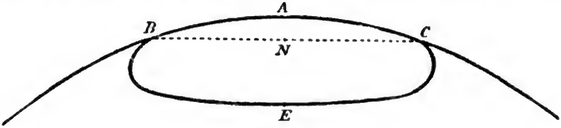

FIG. 127.

$$\cos\frac{\omega}{2} = \frac{h}{\sqrt{2}\,\mathrm{DE}}$$

By this method Magie (*Phil. Mag.*, vol. xxvi. 1888) found the following values for the angle of contact with glass:

Angle zero.	Angle finite.	ω
Ethyl alcohol	Water (?)	small
Methyl alcohol	Acetic acid	20°
Chloroform	Turpentine	17°
Formic acid	Petroleum	26°
Benzine	Ether	16°

Determination of the Surface-tension by Means of Ripples.— The velocity with which waves travel over the surface of a liquid depends on the surface-tension of the liquid. The relation between the velocity and surface-tension may be found as follows: Let Fig. 128 represent the section of a harmonic wave on the surface of the liquid, the undisturbed level of the liquid being xy. If gravity were the only force acting, the increase in vertical pressure at N due to the disturbance produced by the wave would be equal to $g\rho$PN, when ρ is the density of the liquid.

The surface tension will give rise to an additional normal, and therefore approximately vertical, pressure equal per unit area to $\frac{T}{R}$, where R is the radius of curvature of the section of the wave by the plane of the paper; the radius of curvature in the normal plane at right angles to the plane of the paper is infinite. Now if the amplitude of the wave is very small compared with the wave length, the wave curve may be generated by a point fixed to a circle rolling in a straight line; the amplitude is equal to the distance of the point from the centre of the circle, and the wave length is equal to the circumference of the rolling circle. The line xy is the path of the centre of the rolling circle. Now we saw (p. 96) that for such a curve

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* See foot-note on opposite page.

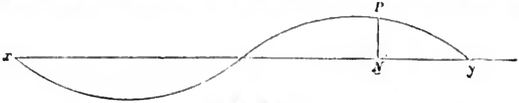

FIG. 128.

$$\frac{1}{R}=\frac{PN}{a^2}$$

where a is the radius of the rolling circle; but if λ is the wave length $2\pi a=\lambda$, so that

$$\frac{1}{R}=\frac{4\pi^2 PN}{\lambda^2}$$

Thus the pressure at N, due both to gravity and surface-tension, is

$$\left(g\rho+\frac{4\pi^2 T}{\lambda^2}\right)PN$$

hence we see that the effects of surface-tension are the same as if gravity were increased by $4\pi^2T/\lambda^2\rho$. Now the velocity of a gravity wave on deep water is the velocity a body would acquire under gravity by falling vertically through a distance $\lambda/4\pi$, where λ is the wave length—*i.e.*, the velocity is $\sqrt{g\lambda/2\pi}$. Hence v, the velocity of a wave propagated under the influence of surface-tension as well as gravity, is given by the equation

$$v=\sqrt{\frac{\lambda}{2\pi}\left\{g+\frac{4\pi^2T}{\lambda^2\rho}\right\}}$$

The velocity of propagation of the wave is thus infinite both when the wave length is zero and when it is infinite; it is proportional to the square root of an expression consisting of the sum of two terms whose product is constant. It follows from a well-known theorem in algebra that the expression will be a minimum when the two terms are equal. Thus the velocity of propagation of the waves will be least when

$$g=\frac{4\pi^2T}{\lambda^2\rho}$$

or when

$$\lambda=2\pi\sqrt{\frac{T}{g\rho}}$$

in this case the velocity is equal to

$$\sqrt{2}\left(\frac{Tg}{\rho}\right)^{\frac{1}{4}}$$

In the case of water, for which $T = 75$,

$$\lambda = 1{\cdot}7 \text{ cm., and } v = 23 \text{ cm./sec.}$$

Hence no waves can travel over the surface of water with a smaller velocity than 23 cm. per second. For any velocity greater than this it is possible to find a wave length λ such that waves of this length will travel with the given velocity. Waves whose lengths are smaller than that corresponding to the minimum velocity are called "ripples," those whose lengths exceed this value "waves." A wave is propagated chiefly by gravity, a ripple chiefly by surface tension.

The velocity of a "wave" increases as the wave length increases, while that of a "ripple" diminishes. Interesting examples of the formation of ripples are furnished by the standing patterns often seen on the surface of running water near an obstacle, such as a stone or a fishing-line. Thus, let AB represent a stone in a stream running from right to left, the disturbance caused by the flow of the water against the stone will give rise to ripples which travel up stream with a velocity depending upon their wave length. Close to the stone the velocity of the water is zero, so that the ripples travel rapidly away from the stone. When, however, we get so far away from the stone, say at P, that the velocity of the water is greater than 23 cm./sec., it is possible to find a ripple of such a wave length that its velocity of propagation over the water is equal to the velocity of the stream, the ripple will be stationary at P, and will form there a pattern of crests and hollows. As the velocity of the water increases as we recede from the stone the ripples which appear stationary must get shorter and shorter in wave length, and thus the crests in the pattern will get nearer and nearer together as we proceed up stream. We see that the condition that the pattern should be formed at all is that the velocity of the stream must exceed 23 cm./sec. Fig. 129 is taken from a photograph of the ripples behind a stone in running water. A similar explanation applies to the pattern in front of a body moving through the liquid.

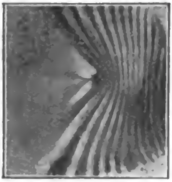

FIG. 129.

Lord Rayleigh was the first (*Phil. Mag.*, xxx. p. 386) successfully to apply the measurement of ripples to the determination of the surface-tension, and his method was used by Dr. Dorsey (*Phil. Mag.*, xliv. p. 369) to determine the surface-tension of a large number of solutions. Lord Rayleigh's method is to generate the ripples by the motion of a glass plate attached to the lower prong of an electrically driven tuning-fork, and dipping into the liquid to be examined. To render the ripples (which for the theory to apply have to be of very small amplitude) visible, light reflected from the surface is brought to a focus near the eye of the observer. On account of the rapidity with which all phases of the waves are presented in succession it is necessary, in order to see the waves distinctly, to use intermittent illumination, the period of the illumination being the same as that of the waves. The illumination can be made intermittent by placing in front of the source of light a piece of tinplate rigidly attached to the prong of a tuning-fork, and so arranged that once on each vibration the light is intercepted by the interposition of the plate. This fork is in unison with the one dipping into the liquid. It is driven electromagnetically, and the intermittent current furnished by this fork is used to excite the vibrations of the dipping fork. By this means the ripples can be distinctly seen, the number between two points at a known distance apart counted, and the wave-length λ determined. If τ is the time of vibration of the fork $v\tau = \lambda$,

and since
$$v^2 = \frac{g\lambda}{2\pi} + \frac{2\pi T}{\lambda\rho}$$

$$\frac{T}{\rho} = \frac{\lambda^3}{2\pi\tau^2} - \frac{g\lambda^2}{4\pi^2}$$

an equation from which T can be determined. The second term in this expression is in these experiments small compared with the first.

Determination of Surface Tension by Oscillations of a Spherical Drop of Liquid.—When the drop is in equilibrium under surface-tension it is spherical; if it is slightly deformed, so as to assume any other form, and then left to itself, the surface-tension will pull it back until it again becomes spherical. When it has reached this state the liquid in the drop is moving, and its inertia will carry the drop through the spherical form. It will continue to depart from this form until the surface-tension is able to overcome the inertia, when it is again pulled back to the spherical form, passes through it and again returns; the drop will thus vibrate about the spherical shape. We can find how the time of vibration depends upon the size of the drop by the method of dimensions, and the problem forms an excellent example of the use of this method. Suppose the drop free from the action of gravity, then t, the time of vibration of the drop, may depend upon a the radius, ρ the density, and S the surface-tension of the liquid; let

$$t = Ca^x\rho^yS^z$$

where C is a numerical constant not depending upon the units of mass, length, or time. The dimensions of the left-hand side are one in time, none in length, and none in mass, which, adopting the usual notation, we denote by $[T]^1\ [L]^0\ [M]^0$; the right-hand side must therefore be of the same dimensions. Now a is of dimensions $[T]^0\ [L]^1\ [M]^0$; ρ, $[T]^0\ [L]^{-3}\ [M]^1$; and S, since it is energy per unit area, $[T]^{-2}\ [L]^0\ [M]^1$; hence the dimensions of $a^x\rho^yS^z$ are, $[T]^{-2z}\ [L]^{-3y+x}\ [M]^{z+y}$. As this is to be of the dimensions of a time, we have

$$-2z = 1, \quad -3y + x = 0, \quad y + z = 0$$

therefore
$$x = \tfrac{3}{2},\ y = \tfrac{1}{2},\ z = -\tfrac{1}{2}$$

So that t, the time of vibration, varies as $\sqrt{\rho a^3/S}$; *i.e.*, it varies as the square root of the mass of the drop divided by the surface-tension; a more complete investigation, involving considerable mathematical analysis, shows that $t = \frac{\pi}{\sqrt{2}}\sqrt{\frac{\rho a^3}{S}}$, where t is the time of the gravest vibration of the drop. The reader can easily calculate the time of vibration of a drop of any size if he remembers that the time of vibration of a drop of water 2·5 cm. in radius is very nearly 1 second. The vibrations of a sphere under surface-tension can easily be followed by the eye if a large spherical drop of water is formed in a mixture of petroleum and bisulphide of carbon of the same density. Lenard (Wiedemann's *Annalen*, xxx. p. 209) applied the oscillation of a drop to determine the surface-tension of a liquid. He determined the time of vibration by taking instantaneous photographs of the drops, and from this time deduced the surface-tension by the aid of the preceding formulæ.

Determination of Surface-tension by the Size of Drops.—The surface-tension is sometimes measured by determining the weight of a drop of the liquid falling from a tube. If we treat the problem as a statical one and suppose that the liquid wets the tube from which it falls, then just on the point of falling the drop below the section *AB* (Fig. 130) is to be regarded as in equilibrium under the surface-tension acting upwards, the weight of the drop acting downwards, the pressure of the air on the surface of the drop acting upwards, and the pressure in the liquid acting downwards across the section *AB*. If a is the radius of the tube, T the surface-tension, then the upward pull is $2\pi a\mathrm{T}$. If we suppose at the instant of falling that the drop is cylindrical at the end of the tube, the pressure in the liquid inside the drop will be greater than the atmospheric pressure by T/a (*see* p. 145). Hence the effect of the atmospheric pressure over the surface of the drop and the fluid pressure across the section AB is a downwards force equal to $\pi a^2\mathrm{T}/a$ or $\pi a\mathrm{T}$. Hence, if w is the weight of the drop we have, equating the upwards and downwards forces,

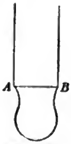

FIG. 130.

$$2\pi a\mathrm{T} = w + \pi a\mathrm{T}\ ;\ \text{or}\ \pi a\mathrm{T} = w.$$

The detachment of the drop is, however, essentially a dynamical phenomenon, and no statical treatment of it can be complete. We should not therefore expect the preceding expression to accord exactly with the results of experiment. Lord Rayleigh[*] finds the relation $3{\cdot}8a\mathrm{T} = w$ to be sufficiently exact for many purposes. Most observers who have used this method seem to have adopted the relation $2\pi a\mathrm{T} = w$, a formula which gives little more than half the true surface-tension; the error comes in by neglecting the change of pressure inside the drop produced by the curvature of its surface.

Wilhelmy's Method.[†]—This consists in measuring the downward pull exerted by a liquid on a thin plate of glass or metal partly immersed in the liquid; the liquid is supposed to wet the plate. The pull can be readily measured by suspending the plate from one of the arms of the balance and observing the additional weight which must be placed in the other scale-pan to balance the pull on the plate when it is partially immersed in the liquid, allowance being made if necessary for the effect of the water displaced. If l is the length of the water-line on the plate, T the surface-tension, then if the liquid wets the plate the downward pull due to surface-tension is $\mathrm{T}l$.

Method of Detachment of a Plate.—Some observers have determined the surface-tension of liquids by measuring the pull required to drag a plate of known area away from the surface. The theory of this method resembles in many respects that by which we determined the thickness of a drop or air bubble (*see* p. 156). Let us take the case of a rectangular plate being pulled away from the surface, and let the figure represent a section by a plane at right angles to the length of the rectangle. Considering the equilibrium of the portion whose section is $EBCF$, and whose length perpendicular to the paper is unity, the horizontal forces acting upon it are: (1) the forces due to surface-tension—*i.e.*, 2T acting from left to right; (2) the atmospheric pressure on the curved surface BC acting from right to left, which is equal to Πd where Π is the atmospheric pressure and d is the height of the lower surface of the plate above the undisturbed level of the liquid; and (3) the fluid pressure acting across the surface EF from left to right. The pressure in the liquid at F is equal to Π, and therefore the resultant fluid pressure across EF is equal to $\Pi d - \frac{1}{2}g\rho d^2$, where ρ is the density of the liquid. Hence, equating the components in the two directions, we have

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[*] Lord Rayleigh, *Phil. Mag.*, 48, p. 321.

[†] Glazebrook and Shaw, *Practical Physics*, ch. vii. § 1.

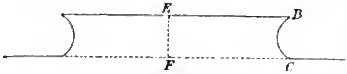

FIG. 131.

$$2\mathrm{T} + \Pi d - \tfrac{1}{2}g\rho d^2 = \Pi d, \text{ or } d^2 = \frac{4\mathrm{T}}{g\rho}$$

Now the fluid pressure just below the surface is less than the atmospheric pressure by $g\rho d$, hence the upward pull P required to detach an area of the plate equal to A is equal to $\mathrm{A}g\rho d$, and substituting for d its value, we find

$$\mathrm{P} = 2\mathrm{A}\sqrt{\mathrm{T}g\rho}.$$

Jaeger's Method.—In this method the least pressure which will force bubbles of air from the narrow orifice of a capillary tube dipping into the liquid is measured. The pressure in a spherical cavity exceeds the pressure outside by $2\mathrm{T}/a$ where a is the radius of the sphere, hence the pressure required to detach the bubble of air exceeds the hydrostatic pressure at the orifice of the tube by a quantity proportional to the surface-tension. This method, which was used by Jaeger, is a very good one when relative and not absolute values of the surface-tension are required; when, for example, we want to find the variation of surface-tension with temperature.

The following are the values of the surface-tension at 0° C., and the temperature coefficients of the surface-tension for some liquids of frequent occurrence. The surface-tension at t° C. is supposed to be equal to $\mathrm{T}_0 - \beta t$.

Liquid	T_0	β
Ether ($C_4H_{10}O$)	19·3	·115
Alcohol (C_2H_6O)	25·3	·087
Benzene (C_6H_6)	30·6	·132
Mercury	527·2	·379
Water	75·8	·152

The surface-tension of salt solutions is generally greater than that of pure water. If T_n is the surface-tension of a solution containing n gramme equivalents per litre, T_w the surface-tension of pure water at the same temperature, Dorsey* has shown that $\mathrm{T}_n = \mathrm{T}_w + \mathrm{R}n$, where R has the following values—NaCl (1·53); KCl (1·71); $\frac{1}{2}(Na_2CO_3)$ (2·00); $\frac{1}{2}(K_2CO_3)$ (1·77); $\frac{1}{2}(ZnSO_4)$ (1·86).

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* Dorsey, *Phil. Mag.*, 44, 1897, p. 369.

On the Effect of Temperature on the Surface-tension of Liquids.—The surface-tension of all liquids diminishes as the temperature increases. This can be shown in the case of water by the following experiment: A pool of water is formed on a horizontal plate of clean metal; powdered sulphur is dusted over the surface of the water and heat applied locally to the under surface of the metal by a fine jet. On the application of the heat the portion of the water immediately over the flame is rapidly swept clear of the sulphur; this is due to the greater tension in the cold liquid outside pulling the sulphur away against the feebler tension in the warmer water.

Eötvos (*Wied. Ann.* 27, p. 448) has pointed out that for many liquids $d(\mathrm{T}v^{\frac{2}{3}})/dt$ is equal to $-2{\cdot}1$, being independent of the nature of the liquid and the temperature; here T is the surface-tension of the liquid, v the "molecular volume"—*i.e.*, the molecular weight divided by the density—and t the temperature. It is clear that, if we assume that $d(\mathrm{T}v^{\frac{2}{3}})/dt$ has this value for a liquid whose density and surface-tension at different temperatures are known, we can determine the molecular weight of the liquid. The method has been applied for this purpose, and some interesting results have been obtained; for example, water is a liquid for which Eötvos' rule does not hold, if we suppose the molecular weight of water to be 18. If, however, we assume the molecular weight of water to be 36—*i.e.*, that each molecule of water has the composition $2H_2O$, then Eötvos' rule is found to hold at temperatures between 100° and 200° C.; below the lower of these temperatures the molecular weight would have to be taken as greater than 36 in order to make Eötvos' rule apply. Hence, Eötvos concluded that the molecules of water, or at any rate the molecules of the surface layers, have the composition $2H_2O$ above 100° C., while below that temperature they have a still more complicated composition.

It follows that if Eötvos' rule is true,

$$\mathrm{T}v^{\frac{2}{3}} = 2{\cdot}1\,(t_1 - t)$$

where t_1 is some constant temperature, which can be determined if we know the value of T and v at any one temperature; t_1 is the temperature at which the surface-tension vanishes, it is therefore a temperature which probably does not differ much from the critical temperature; the values of t_1 for ether, alcohol, water, are roughly about 180°, 295°, 560° C. Their critical temperatures are estimated by Van der Waals to be 190°, 256°, 390° C.

Cooling due to the Stretching of a Film.—Since the surface-tension changes with the temperature, any changes in the area of a film will, as they involve work done by or against surface-tension, be accompanied by thermal changes. We can calculate the amount of these thermal changes if we can imagine a little heat engine which works by the change of surface-tension with temperature. A very simple engine of this kind is as follows: Suppose that we have a rectangular framework on which a film is stretched, and that one of the sides of the framework can move at right angles to its length. Let the mass of the framework and film be so small that it has no appreciable heat capacity. Suppose we have a hot chamber and a cold chamber, maintained respectively at the absolute temperatures θ_1 and θ_2, where θ_1 and θ_2 are so near together that the amount of heat required to raise the body from θ_2 to θ_1 is small compared with the thermal effect due to change of area Let us place the film in the hot chamber, and stretch it so that its area increases by A, then take it out of the hot chamber and place it in the cold one, and allow the film to contract by the amount A; the film has thus recovered its original area. Let it be now placed again in the hot chamber. If the surface-tension of the film when in the cold chamber is greater than when in the hot, then the film when contracting may be made to do more work than was required to stretch it, so that there will be a gain of work on the cycle; the process is plainly reversible, so that the film and its framework and the two chambers constitute a reversible engine. Hence, if H_1 is the heat absorbed in the hot chamber, H_2 that given out in the cold, both being measured in mechanical units, we have by the Second Law of Thermodynamics,

$$\frac{H_1}{\theta_1}=\frac{H_2}{\theta_2}=\frac{H_1-H_2}{\theta_1-\theta_2} \qquad (1)$$

If $T_{\theta_1}, T_{\theta_2}$ are respectively the surface-tensions at the temperatures θ_1 and θ_2, then the work done in stretching the film $=2T_{\theta_1}A$, while the work done by the film when contracting is $2T_{\theta_2}A$, hence the mechanical work gained $=2(T_{\theta_2}-T_{\theta_1})A$. By the principle of the Conservation of Energy the mechanical work gained must equal the difference between the mechanical equivalents of the heat taken from the hot chamber and given up to the cold; hence

$$H_1-H_2=2(T_{\theta_2}-T_{\theta_1})A$$

and from (1)

$$H_1=2\theta_1A\frac{(T_{\theta_2}-T_{\theta_1})}{\theta_1-\theta_2}$$

If β is the temperature coefficient of T, then

$$\beta=\frac{T_{\theta_1}-T_{\theta_2}}{\theta_1-\theta_2}$$

hence

$$H_1=-2\theta_1A\beta$$

Thus H_1 is positive when β is negative, so that when the surface-tension gets less as the temperature increases, heat must be applied to the film to keep the temperature constant when it is extended—*i.e.*, the film if left to itself will cool when pulled out. This is an example of the rule given on page 132 that the temperature change which takes place is such as to make the system stiffer to resist extension. For water β is about T/550, so that the mechanical equivalent of the heat required to keep the temperature constant is about half the work done in stretching the film.

Surface-tension of very thin Films.—The fact that a vertical soap film when allowed to drain shows different colours at different places and is yet in equilibrium shows that the thickness of the film may vary within wide limits without any substantial change in the surface-tension. The connection between the thickness of the film and the surface-tension was investigated by Rücker and Reinold.[*] The method used is represented diagrammatically in Fig. 132. Two cylindrical films were balanced against each other, and one of them was kept thick by passing an electric current up it; this keeps the film from draining, the other film was allowed to drain, and a difference of surface-tension was indicated by a bulging of one of the cylinders and a shrivelling of the other. When films are first formed the value of their surface-tension is very irregular; but Rücker and Reinold found that, if they were allowed to get into a steady state, then a direct comparison of the surface-tension over a range of thickness extending from 1350 $\mu.\mu$ ($\mu.\mu$ is 10^{-7} cm.) down to the stage of extreme tenuity, when the film shows the black of the first order of Newton's scale of colour, showed no appreciable change in surface-tension, although, had the difference amounted to as much as one-half per cent., Reinold and Rücker believed they could have detected it. A large number of determinations of the thickness of the black films were made, some by determining the electrical resistance and then deducing the thickness, on the assumption that the specific resistance is the same as for the liquid in bulk, others by determining the retardation which a beam of light suffers on passing through the film, and assuming the refraction index to be that of the liquid in mass: all these determinations gave for the thickness of the black films a constant value about 12 $\mu.\mu$. At first sight it appears as if the surface-tension suffered no change until the thickness is less than 12 $\mu.\mu$. The authors have shown, however, that this is not the right interpretation of their results, for they find that the black and coloured parts of the film are separated by a sharp line showing that there is a discontinuity in the thickness. In extreme cases the rest of the film may be as much as 250 times thicker than the black part with which it is in contact. The section of a film showing a black part is of the kind shown in Fig. 133. The stability of the film shows that the tension in the thin part is equal to that in the thick. It is remarkable that in these films there are never any parts of the film with a thickness anywhere between 12 $\mu.\mu$. and something between 45 and 95 $\mu.\mu$.; films whose thicknesses are within this range are unstable. This is what would occur if the surface-tension first begins to diminish at the upper limit of the unstable thickness, and after diminishing for some time, then begins to increase as the thickness of the film gets less, until at 12 $\mu.\mu$. it has regained its original value; after this it increases for some time, and then diminishes indefinitely as the thickness of the film gets smaller and smaller. The changes in surface-tension are represented graphically by the curve in Fig. 134, where the ordinates represent the surface-tension and the abscissæ the thickness of the film. For suppose we have a film thinning, it

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[*] Rücker and Reinold, *Phil. Trans.* 177, part ii. p. 627, 1886.

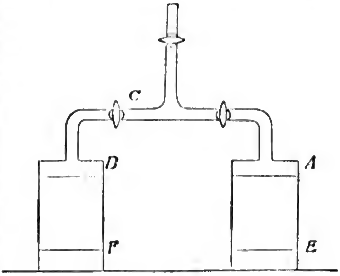

FIG. 132.

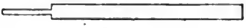

FIG. 133.

will be in equilibrium until the upper part gets the thickness corresponding to the point P on the curve; as the tension now gets less than in the thicker part of the film, the thicker parts pull the thin part away, and would certainly break it, were it not that after the film gets thinner than at R the tension increases until, when the film reaches the thickness corresponding to Q, the tension is the same as in the thick film, and there is equilibrium between the thick and the thin pieces of the film. This equilibrium would be stable, for if the film were to get thinner the tension would get greater, and the film would contract and thicken again, while if it got thicker the tension would fall and the film would be pulled out until it regained its original thickness. Thus all the films which are in contact with thick films must have the constant thickness corresponding to Q. The equilibrium at S, when the tension has the same value as at Q, is unstable, for any extension of the film lowers the tension, and thus makes the film yield more readily to the extension. The region between R and P is unstable, so is that between T and O. The region TR would be stable, but would be very difficult to realise. If we start with a thick film and allow it to thin, the only films of thickness less than that at P which will endure will be those whose thickness is constant and equal to the thickness at Q. Johannot (*Phil. Mag.*, 47, p. 501, 1899) has recently shown that a black film of oleate of soda may consist of two portions, one having a thickness of 12 $\mu.\mu$, the other of 6 $\mu.\mu$. In this case there must be another dip between S and R in the curve representing the relation between surface-tension and thickness.

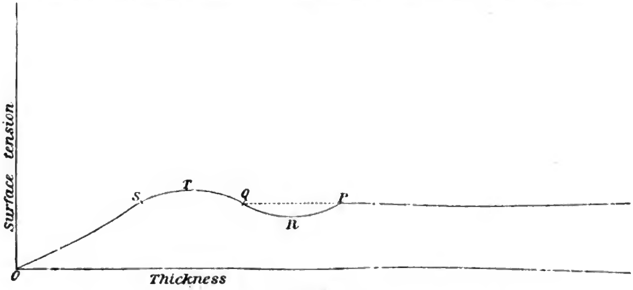

FIG. 134.

Vapour Pressure over a Curved Surface.—Lord Kelvin was the first to show that in consequence of surface-tension the vapour pressure in equilibrium with a curved surface is not the same as the pressure of the vapour in equilibrium with a flat one. We can see from very general considerations that this must be the case, for when water evaporates from a flat surface there is no change in the area of the surface and therefore no change in the potential energy due to surface-tension; in the case of a curved surface, however, such as a spherical drop, when water evaporates there will be a diminution in the area of the surface and therefore a diminution in the potential energy due to surface-tension. Thus the surface-tension will promote evaporation in this case, as evaporation is accompanied by a diminution in the potential energy. Thus evaporation will go on further from a spherical drop than from a plane surface; that is, the pressure of the water vapour in equilibrium with the spherical drop is greater than for the plane area.

Lord Kelvin's determination of the effect of curvature on the vapour pressure is as follows: Let a fine capillary tube be placed in a liquid, let the liquid rise to A in the tube, and let B be the level of the liquid in the outer vessel. Then there must be a state of equilibrium between the liquid and its vapour both at **A** and B, otherwise evaporation or condensation would go on and the system would not attain a steady state. Let p p' be the pressures of the vapour of the liquid at B and A respectively, h the height of A above B,

$$p = p' + \text{pressure due to a column of vapour whose height is } h$$

$$= p' + g\sigma h, \qquad (1)$$

where σ is the density of the vapour. If r is the radius of the surface of the liquid at **A**, then T being the surface-tension,

$$\frac{2T}{r} = \text{difference of pressure on the two sides of the meniscus.}$$

Now the pressure on the liquid side of the meniscus is equal to $\Pi - g\rho h$ where ρ is the density of the liquid and Π the pressure at the level of the liquid surface in the outer vessel; the pressure on the vapour side of the meniscus is $\Pi - g\sigma h$; thus the difference of pressures is equal to $g(\rho - \sigma)h$, so that

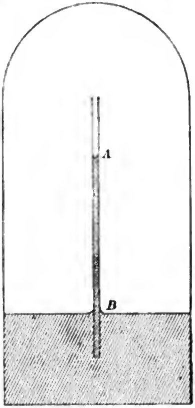

Fig. 135.

$$\frac{2T}{r} = g(\rho - \sigma)h^{*}$$

or

$$g\sigma h = \frac{2T}{r}\frac{\sigma}{\rho - \sigma}$$

Hence by equation (1)

$$p' = p - \frac{2T}{r}\frac{\sigma}{\rho - \sigma}\dagger$$

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* In the investigation of the capillary ascent in tubes given on p. 141, σ is neglected in comparison with ρ.

† The formula in the text gives the value for $p' - p$ when this is small compared with p; the general equation for p' may be proved to be (neglecting σ in comparison with ρ)

$$\log\frac{p'}{p} = \frac{-2T}{r\rho}\cdot\frac{1}{R\theta}$$

where θ is the absolute temperature and R the constant in the equation for a perfect gas—*i.e.*, $pv = R\theta$.

hence the equilibrium vapour pressure over the concave hemispherical surface is less than that over a plane surface at the same temperature by $2T\sigma/(\rho-\sigma)r$. We may write this as $\omega\sigma/(\rho-\sigma)$ where ω is the amount by which the pressure below the curved surface is less than that below the plane. If the shape of the liquid surface had been convex, like that of a dewdrop, instead of concave, the pressure below the curved surface of liquid would be greater than that in the plane surface instead of being less, and the pressure of the water vapour over the surface would be greater than that over a plane surface. It can be shown that if an external pressure ω were applied to a plane surface the vapour pressure would be increased by $\omega\sigma/\rho$ (see J. J. Thomson, *Applications of Dynamics* p. 171). Unless the drops are exceedingly small, the effect of curvature on the vapour pressure is inappreciable; thus if the radius of the drop of water is one-thousandth part of a millimetre the change in the vapour pressure only amounts to about one part in nine hundred. As the effect is inversely proportional to the radius, it increases rapidly as the size of the drop diminishes, and for a drop 1 $\mu.\mu$ in radius the vapour pressure over the drop when in equilibrium would be more than double that over a plane surface. Thus a drop of this size would evaporate rapidly in an atmosphere from which water would condense on a plane surface. This has a very important connection with the phenomena attending the formation of rain and fog by the precipitation of water vapour. Suppose that a drop of water had to grow from an indefinitely small drop by precipitation of water vapour on its surface; since the vapour pressure in equilibrium with a very small drop is much

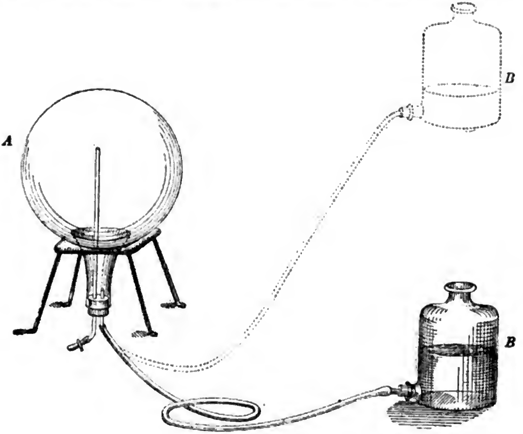

FIG. 136.

greater than the normal, the drop, unless placed in a space in which the water vapour is in a very supersaturated condition, will evaporate and diminish in size instead of being the seat of condensation and increasing in radius. Thus these small drops would be unstable and would quickly disappear. Hence it would seem as if this would be an insuperable difficulty to the formation of drops of rain or cloud if these drops have to pass through an initial stage in which their size is very small. Aitken has shown that as a matter of fact these drops are not formed under ordinary conditions when water and water vapour alone are present, even though the vapour is considerably oversaturated, and that for the formation of rain and fog the presence of dust is necessary. As the water is deposited around the particles of dust, the drops thus commence with a finite radius, and so avoid the difficulties connected with their early stages. The effect of dust on the formation of cloud can be shown very easily by the following experiment. *A* and *B* are two vessels connected with each other by a flexible pipe; when *B* is at the upper level indicated in the diagram the globe *A* is partly filled with water; if the vessel *B* is lowered the water runs out of *A*, the volume of the gas in *A* increases, and the cooling caused by the expansion causes the region to be oversaturated with water vapour. If *A* is filled with the ordinary dusty air from a room, a cloud is formed in *A* whenever *B* is lowered; this cloud falls into the water, carrying some dust with it; on repeating the process a second time more dust is carried down, and so by continued expansions the air can be made dust free. We find that, after we have made a considerable number of expansions, the cloud ceases to be formed when the expansion takes place; that the absence of the cloud is due to the absence of dust can be proved by admitting a little dust through the tube; on making the gas expand again a cloud is at once formed.

It was supposed for some time that without dust no clouds could be formed, but it has been shown by C. T. R. Wilson that gaseous ions can act as nuclei for cloudy condensation if the supersaturation exceeds a certain value, and he has also shown that if perfectly dust-free air has its volume *suddenly increased* 1·4 time a dense cloud is produced. Though dust is not absolutely essential for the formation of clouds, yet the conditions under which clouds can be formed without dust are very exceptional, inasmuch as they require a very considerable degree of supersaturation.

Movement of Camphor on Water.—If a piece of camphor is scraped and the shavings allowed to fall on a clear water surface they dance about with great vigour. This, as Marangoni has shown, is due to the camphor dissolving in the water, the solution having a smaller surface-tension than pure water; thus each little patch of surface round a particle of camphor is surrounded by a film having a stronger surface-tension than its own, it will therefore be pulled out and the surface of the water near the bit of camphor set in motion. For the movements to take place the surface-tension of the water surface must be greater than that of the camphor solution; if the surface is greasy the surface-tension is less than that of pure water, and may be so much reduced that it is no longer sufficient to produce the camphor movements. Lord Rayleigh has measured the thickness of the thinnest film of oil which will prevent the motion of the camphor; the thickness was determined by weighing a drop of oil which was allowed to spread over a known area. He found that to stop the camphor movements (which involved a reduction of the surface-tension by about 28 per cent.) a layer of oil 2 $\mu\mu$ thick was required (1 $\mu\mu = 10^{-7}$ cm.), and that with thinner films the movements were still perceptible. This thickness is small compared with 12 $\mu\mu$ the thickness found by Rücker and Reinold for black films, but it must be remembered that the surface which stops the camphor movements is still far from acting as a surface of oil; the surface-tension, though less than that of water, is greater than that of oil. The manner in which the tension of a contaminated water surface varies with the amount of contamination has been investigated by Miss Pockels and also by Lord Rayleigh (*Phil. Mag.*, 48, p. 321). Miss Pockels determined the surface-tension by measuring the force required to detach a disc of known area from the surface; Lord Rayleigh used Wilhelmy's method. The amount of contamination was varied by confining the greased surface between strips of glass or metal dipping into the water; by pulling these apart the area of the greased surface was increased and therefore the thickness of it diminished, while by pushing them together the thickness could be increased.

The way in which the surface-tension is affected by the thickness of the layer of grease is shown by the curve (Fig. 137) given by Lord Rayleigh.

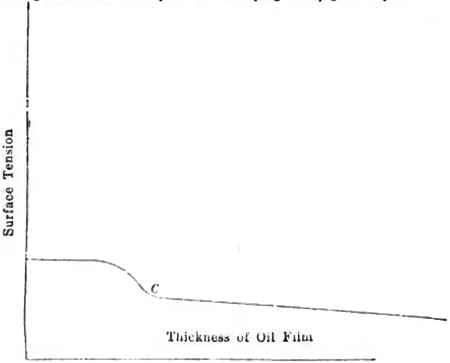

FIG. 137.

In this curve the ordinates are the values of the surface-tension, the abscissæ the thicknesses of the oil film; both of these are on an arbitrary scale. It will be seen that no change in the surface-tension occurs until the thickness of the oil film exceeds a certain value (about $1\mu.\mu$); at this stage the surface-tension begins to fall rapidly and continues to do so until it reaches the thickness corresponding to the point C (about $2.\mu.\mu.$); this is called the camphor point, being the thickness required to stop the movements of the camphor particles. After passing this point the variation of the surfac-tension with the thickness of the film becomes much less rapid. Lord Rayleigh gives reasons for thinking that the thickness $1\mu.\mu$ is equal to the diameter of a molecule of oil.

Thus, when the amount of contamination is between the limits corresponding to a thickness of the surface layer between 1 $\mu.\mu$ and the smallest thickness required to give the surface-tension of oil, any diminution in the contamination such as would be produced by an extension of the surface would result in an increase in the surface-tension. This is a principle of great importance; it seems first to have been clearly stated by Marangoni. Suppose we push a strip of metal along a surface in this condition, the metal will heap up the grease in front and scrape the surface behind, thus the surface-tension behind the strip will be greater than that in front, so that the strip will be pulled back; there will thus be a force resisting the motion of the strip due to the variation of the surface-tension. This is Marangoni's explanation of the phenomenon of superficial viscosity discovered by Plateau. Plateau found that if a vibrating body such as a compass-needle was disturbed from its position of equilibrium and then allowed to return to it (1) with its surface buried beneath the surface of the liquid, (2) with its face on the surface of the liquid, then with certain liquids, of which water was one, the time taken in the second case is considerably greater than that in the first. We see that it must be so if the surface of the liquid is contaminated by a foreign substance which lowers its surface-tension.

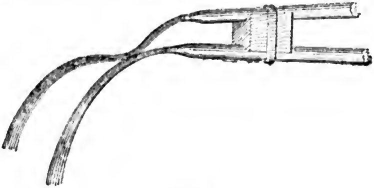

FIG. 138.

Calming of Waves by Oil.—Similar considerations will explain the action of oil in stilling troubled waters. Let us suppose that the wind acts on a portion of a contaminated surface, blowing it forward; the motion of the surface film will make the liquid behind the patch cleaner and therefore increase its surface-tension, while it will heap up the oil in front and so diminish the surface-tension; thus the pull back will be greater than the pull forward, and the motion of the surface will be retarded in a way that could not occur if it were perfectly clean. The oiled surface acts so as to check any relative motion of the various parts of the surface layer and so prevents any heaping up of the water. It is these heaps of water which, under the action of the wind, develop into a high sea; the oil acts not so much by smoothing them down after they have grown as by stifling them at their birth.

A contaminated surface has a power of self-adjustment by which the surface-tension can adjust itself within fairly wide limits; a film of such a liquid can thus, as Lord Rayleigh points out, adjust itself so as to be in equilibrium under circumstances when a film of a pure liquid would have to break. Thus, to take the case of a vertical film, if the surface-tension were absolutely constant, as it is in the case of a pure liquid when the film is not too thin, this film would break, since there would be nothing to balance the weight of the film. If, however, the film were dirty, a very slight adjustment of the amount of dirt at different parts of the surface would be sufficient to produce a distribution of surface-tension which would ensure equilibrium. It is probably on this account that films to be durable have to be made of a mixture of substances, such as soap and water.

Collision of Drops.—If a jet of water be turned nearly vertically upwards the drops into which it breaks will collide with each other; if the water is clean the drops will rebound from each other after a collision, but if a little soap or oil is added to the water, or if an electrified rod is held near the jet, the drops when they strike will coalesce instead of rebounding, and in consequence will grow to a much larger size. This can be made very evident by allowing the drops to fall on a metal plate; the change in the tone of the sound caused by the drops striking against the plate when an electrified rod is held near the jet is very remarkable.

The same thing can be shown with two colliding streams. If two streams of pure water strike against each other in dust-free air, as in Fig. 138, they will rebound; if an electrified rod is held near, however, they coalesce.

CHAPTER XV.

LAPLACE'S THEORY OF CAPILLARITY.

CONTENTS.—Intrinsic Pressure in a Fluid—Work required to move a Particle from the Inside to the Outside of a Liquid—Work required to produce a new Liquid Surface—Effect of Curvature of surface—Thickness at which Surface-tension changes effect of abruptness of transition between two Liquids in contact.

LAPLACE'S investigations on surface-tension throw so much light on this subject, as well as on the constitution of liquids and gases, that no account of the phenomena associated with surface-tension would be complete without an attempt to give a sketch of his theory. Laplace started with the assumption that the forces between two molecules of a liquid, although very intense when the distance between the molecules is very small, diminish so rapidly when this distance increases that they may be taken as vanishing when the distance between the molecules exceeds a certain value c; c is called the range of molecular action. We shall find that we can obtain an explanation of many surface-tension phenomena even although we do not know the law of force between the molecules. Let the attraction of an infinite flat plate of the fluid bounded by a plane surface on a mass m at a point at a distance z above the surface be $m\sigma\psi(z)$, where σ is the density of the fluid; in accordance with our hypothesis $\psi(z)$ vanishes when z is greater than c. It is evident, too, that $m\sigma\psi(z)$ will be the attraction at a point on the axis of any disc with a flat face whose thickness is greater than c and whose diameter is greater than $2c$.

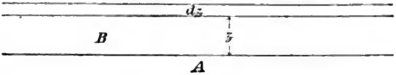

FIG. 139.

Suppose we imagine a fluid divided into two portions A and B by a plane; let us find the pull exerted on B by A. Divide B up into thin layers whose thickness is dz; then if z is the height of one of these layers above the surface of separation the force on unit area of the layer is equal to $\sigma\psi(z)\sigma dz$;

hence the pull of A on B per unit area is equal to $\sigma^2\int_0^{\infty}\psi(z)dz$,

which, since $\psi(z)$ vanishes when $z > c$, is the same as $\sigma^2\int_0^{c}\psi(z)dz$.

This pull between the portions A and B is supposed to be balanced by a pressure called the "intrinsic pressure," which we shall denote by K. K then is equal to

$$\sigma^2\int^{\infty}\psi(z)dz$$

We shall find that the phenomena of capillarity require us to suppose that, in the case of water, the intrinsic pressure is very large, amounting on the lowest estimate to several thousand atmospheres. We may remark in passing that the intrinsic pressure plays a very important part in Van der Waals' Theory of the Continuity of the Liquid and Gaseous States; it is the term a/v^2 which occurs in his well-known equation

$$\left(p + \frac{a}{v^2}\right)(v - b) = \mathrm{RT} \qquad (\textit{see } \text{p. } 129)$$

We see, too, at once from the preceding investigation that K is equal to the tensile strength of the liquid, so that if the common supposition that liquids are as "weak as water," and can only bear very small tensile stresses without rupture, were true, Laplace's theory, which, as we have seen, requires liquids to possess great tensile strength, would break down at the outset. We have seen, however, p. 122, that the rupture of liquids under ordinary conditions gives no evidence as to the real tensile strength of the liquids, for it was shown that when water and other liquids are carefully deprived of gas bubbles—in fact, when they are not broken before the tension is applied—they can stand a tension of a great many atmospheres without rupture; thus on this point the properties of liquids are in accordance with Laplace's theory.

There is another interpretation of K given by Dupré which enables us to form an estimate of its value. Consider a film of thickness Δ (where Δ is small compared with c) at the top of the liquid; the work required to pull unit of area of this film off the liquid and remove it out of the sphere of its attraction is evidently

$$\sigma^2\Delta\int_0^\infty \psi(z)dz \text{ or } \mathrm{K}\Delta$$

Thus the work required to remove unit volume of the liquid and scatter it through space in the form of thin plates whose thickness is small compared with the range of molecular attraction is K. Now the work required to take one of these films and still further disintegrate it until each molecule is out of the sphere of action of the others will be small compared with the work required to tear the film off the surface of the liquid; hence K is the work required to disintegrate unit volumes of the liquid until its molecules are so far apart that they no longer exert any attraction on each other; in other words, it is the work required to vaporise unit volume of the gas. In the case of water at atmospheric temperature this is about 600 thermal units or $600 \times 4{\cdot}2 \times 10^7 = 25{\cdot}2 \times 10^9$ mechanical units; or since an atmosphere expressed in these units is 10^6 this would make K equal to about 25 000 atmospheres [*]

Work required to move a Particle from the Inside to the Outside of a Fluid.—Consider the force on a particle P at a depth z below the surface; the force due to the stratum of fluid above P will be balanced by the attraction of the stratum of thickness z below P; thus the force acting on P will be that due to a slab of liquid on a particle at a distance z above its surface—*i.e.*, $m\sigma\psi(z)$. Hence the work done in bringing the particle to the surface is

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[*] Van der Waals gives the following value of K deduced from his equation: water 10,500–10,700, ether 1300–1430, alcohol 2100–2100, carbon bisulphide 2900–2890 atmospheres.

$$m\int_0^\infty \sigma\psi(z)dz = m(\mathrm{K}/\sigma);$$

as an equal amount of work will be required to take the particle from the surface out of the range of molecular attraction, the total amount of work required is thus $2m(\mathrm{K}/\sigma)$.

Hence, if a particle moving with a velocity v towards the surface starts from a depth greater than c it cannot cross the surface unless

$$\tfrac{1}{2}mv^2 > \frac{2m\mathrm{K}}{\sigma} \text{ or } v^2 > \frac{4\mathrm{K}}{\sigma} .$$

In the case of water, for which $\sigma = 1$ and K on the preceding estimate is 25,000 atmospheres or $2{\cdot}5 \times 10^{10}$, we see that a particle would not cross the surface unless its velocity were greater than $3{\cdot}2 \times 10^5$. The average velocity of a molecule of water vapour at 0° C. is about 6×10^4, so that if the water contained molecules of water vapour it would only be those possessing a velocity considerably greater than the mean velocity, which would be able to escape across the surface.

Work required to produce a new Liquid Surface.—Let us consider the amount of work required to separate the two portions *A* and *B* into which a plane *C* divides the liquid. Dividing *B* up, as before, into slices parallel to the interface, then the work done in removing the slice, whose thickness is dz and whose height above the plane is z, is per unit of area equal to

$$\sigma^2 dz\int_z^\infty \psi(x)dx = \sigma^2 dz v, \text{if } v = \int_z^\infty \psi(x)dx$$

hence the work required to remove the whole of the liquid *B* standing on unit area away from *A* is

$$\int_0^\infty \sigma^2 v dz;$$

integrating this by parts we see that it is equal to

$$\Big[\sigma^2 z v\Big]_0^\infty - \int_0^\infty \sigma^2 z\frac{dv}{dz}dz$$

Now the term within brackets vanishes at both limits, and $\dfrac{dv}{dz} = -\psi(z)$,

hence the work required is

$$\sigma^2\int_0^\infty z\psi(z)dz$$

For this amount of work we have got 2 units of area of new surface, hence the energy corresponding to each unit of area (*i.e.*, the surface-tension), which we shall denote by T is given by the equation

$$\mathrm{T} = \tfrac{1}{2}\sigma^2\int_0^\infty z\psi(z)dz \qquad (1)$$

Young, at the beginning of the century, showed how from T and K it was possible to calculate the range of molecular forces. He did this by assuming a particular value for the force, but his argument is applicable even when we leave the force undetermined.

If $\psi(z)$ is always positive, then, since c is the greatest value of z for which $\psi(z)$ has a finite value, we see from equation (1) that

$$\mathrm{T} < \tfrac{1}{2}\sigma^2 c\int_0^\infty \psi(z)dz$$

$$< \tfrac{1}{2}c\mathrm{K}$$

hence
$$c > \frac{2\mathrm{T}}{\mathrm{K}}.$$

For water $\mathrm{T} = 75$, and if we take $\mathrm{K} = 25{,}000$ atmospheres $= 2.5 \times 10^{10}$, then the above relation shows that $c > 6 \times 10^{-9}$. In this way we can get an inferior limit to the range of molecular action. This method, which was given by Young, was the first attempt to estimate this quantity, and it seems to have been quite overlooked until attention was recently called to it by Lord Rayleigh.

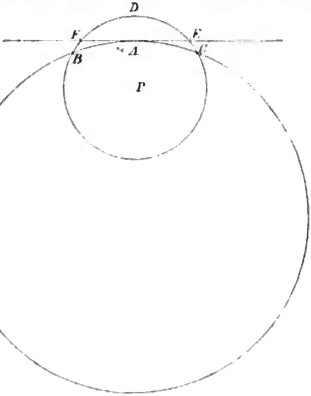

FIG. 140.

It is instructive to consider another way of finding the expression for the surface-tension. Consider a point P inside a liquid sphere (Fig. 140). Then, if P is at depth d, below the surface, greater than c, the forces acting on it, due to the attraction of the surrounding molecules, are in equilibrium; if the distance of P below the surface is less than c, then to find the force on P describe a sphere with radius c and centre P, and the force on *P*, acting towards the centre of the larger sphere, will be equal to the attraction which would be exerted on *P* by a quantity of the fluid placed so as to fill *BACD*, the portion of the sphere whose centre is *P*, which is outside the larger sphere. This portion may be regarded as consisting of two parts—(1) the portion above the tangent plane at *A*, the point on the large sphere nearest to *P*, and (2) the lenticular portion between this plane and the sphere. Now the attraction of the portion above the tangent plane is the same as that of a slab of the liquid extending to infinity and having the tangent plane for its lower face, for the portions of liquid which have to be added to the volume *ADEF* to make up this slab are at a greater distance from *P* than *c*, and so do not exert any attraction on matter at *P*. Thus, if $AP = z$, the attraction of *AFDE* on unit mass at *P*, using the previous notation, is $\sigma\psi(z)$; the attraction of the lenticular portion at *P* can be shown to be $\frac{\sigma z}{\mathrm{R}}\psi(z)$ where R is the radius of the liquid sphere. Hence the total force at *P* acting on unit mass in the direction *AP* is equal to

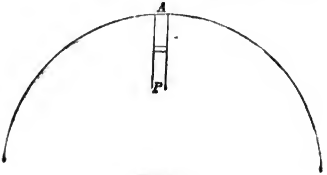

FIG. 141.

$$\sigma\psi(z) + \frac{\sigma z}{\mathrm{R}}\psi(z) \qquad (3)$$

Consider now the equilibrium of a thin cylinder of the fluid, the axis of the cylinder being *PA*; divide this cylinder up into thin discs, then if dz is the thickness of a disc, z its distance from *A* and a the area of the cross-section of the cylinder, the force acting on this disc is equal to

$$\left\{\sigma^2\psi(z) + \frac{\sigma^2 z}{\mathrm{R}}\psi(z)\right\} a\,dz$$

This force has to be balanced by the excess of pressure on the lower face of the disc over that on the upper face; this excess of pressure is, if p represents the pressure, equal to $a\frac{dp}{dz}dz$;

hence, equating this to the force acting on the disc, we get

$$\frac{dp}{dz} = \sigma^2\psi(z) + \sigma^2\frac{z}{\mathrm{R}}\psi(z)$$

Thus the excess of pressure at a point at a distance c, below *A* over the pressure at *A* is equal to

$$\int_0^c \sigma^2\psi(z)dz + \int_0^c \sigma^2\frac{z}{\mathrm{R}}\psi(z)dz$$

or with our previous notation $\quad \mathrm{K} + \frac{2\mathrm{T}}{\mathrm{R}}$.

The pressure has the same value at all points whose depth below the surface is greater than c. The term 2T/R represents the excess of pressure due to the curvature of the surface; we obtained the same value by a different process on p. 145. If the surface of the liquid sphere had been concave instead of convex, an inspection of the figure shows that to obtain the force on P we should have to *subtract* the attraction due to the lenticular portion from the attraction due to the portion ADE instead of adding it; this would make the pressure at a point in the mass of the fluid less than that at a point in the fluid but close to the surface by 2T/R.

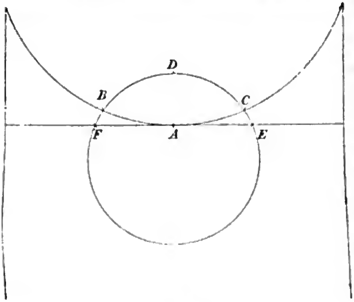

FIG. 142.

Thickness at which the Surface-tension changes.—We can determine the point at which the surface-tension begins to change by finding the change of pressure which takes place as we cross a thin film. Let Fig. 143 represent the section of such a film, bounded by spheres; if the thickness of the film is small, the radii of these spheres may be taken as approximately equal. Let P be a point in the film, ABP a line at right angles to both surfaces, then the investigation just given shows that if $AP = z$, $BP = z'$, the force on unit mass at P is equal to

$$\sigma\psi(z) + \sigma\frac{z}{\mathrm{R}}\psi(z) - \{\sigma\psi(z') - \sigma\frac{z'}{\mathrm{R}}\psi(z')\}$$

when R is the radius of one of the films. We see, too, from the last paragraph that the pressure at B must be greater than that at A by

$$\int_0^t \{\sigma^2\psi(z) + \frac{\sigma^2 z}{\mathrm{R}}\psi(z)\}dz - \int_0^t \{\sigma^2\psi(z') - \frac{\sigma^2 z'}{\mathrm{R}}\psi(z')\}dz' = \frac{2\sigma^2}{\mathrm{R}}\int_0^t z\psi(z)dz$$

where t is equal to AB, the thickness of the film. Hence, from the formula (p. 145) for the difference of pressure inside and outside a soap bubble, we may regard

$$\frac{\sigma^2}{2}\int_0^t z\psi(z)dz$$

as the surface-tension of a film of thickness t. Since $\psi(z)$ vanishes when z is greater than c, the surface-tension will reach a constant value when t is as great as c; hence c, the range of molecular action, is the thickness of a film when the surface-tension begins to fall off. When t is less than c we see from the preceding expression that, T being the surface-tension,

$$\frac{d\mathrm{T}}{dt}=\frac{\sigma^2}{2}t\psi(t)$$

Now if T is represented by a curve like Fig. 134, $d\mathrm{T}/dt$ is zero down to P, positive from P to R, negative from R to T, and positive again for all thinner films; hence, since the force of a slab is attractive when ψ is positive, repulsive when ψ is negative, this would imply, on Laplace's theory, that the molecular forces due to a slab of liquid at a point outside are at first attractions; then, as the point gets nearer the slab, they change to repulsions, and change again to attractions as the point approaches still nearer to the slab. If t is so small that $\psi(t)$ can be regarded as constant, we see that T will vary as t^2, so that ultimately the surface-tension will diminish very rapidly as the film gets thinner.

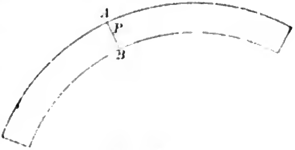

Fig. 143.

On the Effect of the Abruptness of Transition between two Liquids on the Surface-tension of their Interface.—Laplace assumed that the range of molecular forces was the same for all bodies, and that at equal distances the force was proportional to the density of the substance. This implies that the function $\psi(z)$ is the same for all bodies. This hypothesis is certainly not general enough to cover all the facts; it is probably, however, sufficiently general to give the broad outlines of capillary phenomena. Let us calculate on this hypothesis the surface-tension between two fluids A and B. Let σ_1 σ_2 be the densities of these fluids; then to separate a sphere whose area is S from the liquid A requires the expenditure of work equal to

$$\tfrac{1}{2}\mathrm{S}\sigma_1^2\int_0^\infty z\psi(z)dz \qquad (\textit{see } \text{p. } 175)$$

Let us make a spherical hole of equal size in B. To do this will require the expenditure of an amount of work equal to

$$\tfrac{1}{2}\mathrm{S}\sigma_2^2\int_0^\infty z\psi(z)dz$$

Let us place the sphere A in the hole in B, and let the fluids come into contact under their molecular forces; during this process the amount of work done by these forces is

$$\mathrm{S}\sigma_1\sigma_2\int^\infty z\psi(z)dz$$

Hence the total expenditure of work required to produce an area S of interface of A and B is

$$\tfrac{1}{2}S\sigma_1^2\int_0^\infty z\psi(z)dz + \tfrac{1}{2}S\sigma_2^2\int_0^\infty z\psi(z)dz - S\sigma_1\sigma_2\int_0^\infty z\psi(z)dz$$

$$= \tfrac{1}{2}S(\sigma_1 - \sigma_2)^2\int_0^\infty z\psi(z)dz$$

But this work is by definition equal to $T_{AB}S$ where T_{AB} is the surface-tension between A and B; hence we see that $T_{AB} = (\sigma_1 - \sigma_2)^2C$, where

$$C = \tfrac{1}{2}\int_0^\infty z\psi(z)dz$$

is a constant for all substances. This result is not a complete representation of the surface-tension, for if it were there would always be surface-tension between liquids of different densities, so that two such liquids could not mix; it would also require that the surface-tension between fluids of equal density should be zero, and that

$$\sqrt{T_{AB}} = \sqrt{T_{AC}} + \sqrt{T_{CB}}$$

where T_{AB}, T_{AC}, and T_{CB} are respectively the surface-tensions between fluids A and B, A and C, and B and C respectively. None of these results are in accordance with experiment. Let us, however, on the assumption that the surface-tension is represented by an expression of this kind, calculate (following Lord Rayleigh) the effect of making the transition between A and B more gradual; we can do this by supposing that we have between A and B a layer of a third fluid C whose density is the arithmetical mean between the densities of A and B; then $T_{AC} = \frac{1}{4}T_{AB} = T_{CB}$. Hence, though now we have two surfaces of separation instead of one, the energy per unit area of each is only one quarter of that of unit area of the original surface; hence the total energy due to surface-tension is only one half of the energy when the transition was more abrupt. By making the transition between A and B still more gradual by interposing n liquids whose densities are in arithmetical progress, we reduce the energy due to surface-tension to $1(n+1)$ of its original value. Thus we conclude that any diminution in the abruptness will diminish the energy due to surface tension. This result may have important bearings on the nature of chemical action between the surface layers of liquids in contact, for if a layer of a chemical compound of A and B were interposed between A and B the transition between A and B would be less abrupt than if they were directly in contact, and therefore the potential energy, as far as it results from surface-tension, would be less. Chemical combination between A and B would result in a diminution of this potential energy. Now anything that tends to increase the diminution in potential energy resulting from the chemical combination promotes the combination; the forces that give rise to surface-tension would, therefore, tend to promote the chemical combination. Thus, in the chemical combination between thin layers of liquid there is a factor present which is absent or insignificant in the case of liquids in bulk, and we may expect that chemical combination between thin layers of liquids might take place even though it were absent in ordinary cases.

Similar considerations would lead us to expect changes in the strength of a solution near the surface whenever the surface-tension of the solution depends upon its strength: if the surface-tension increased with the strength there would be a tendency for the salt to leave the surface layers, while if the surface-tension diminished as the strength of the solution increased the salt would tend to get to the surface, so that the surface layers would be stronger solutions than the bulk of the liquid. The concentration or dilution of the surface layers would go on until the gradient of the osmotic pressures resulting from the variation in the strengths of different layers is so great that the tendency to make the pressure equal just balances the effects due to surface-tension.

CHAPTER XVI.

DIFFUSION OF LIQUIDS.

CONTENTS.—General Law of Diffusion—Methods of determining the Co efficient of Diffusion—Diffusion through Membranes. Osmosis—Osmotic Pressure—Vapour Pressure of a Solution—Elevation of the Boiling-point of Solutions—Depression of the Freezing-point—Dissociation of Electrolytes.

IF two liquids are left in contact with each other and are free from the action of external forces, then if they can mix in any proportion they will of themselves go on mixing until the whole mass is uniform in composition. This process may be illustrated by taking a vertical glass tube and filling the lower part with a strong solution of a coloured salt, such as copper sulphate. On the top of this clear water is poured very slowly and carefully, so as not to give rise to any currents in the liquid. The coloured part will at first be separated from the clear by a sharply marked surface, but if the vessel is left to itself it will be found that the upper part will become coloured, the colour getting fainter towards the top, while the colour in the lower part of the tube will become fainter than it was originally. This change in colour will go on until ultimately the whole of the tube is of a uniform colour. There is thus a gradual transference of the salt from the places where the solution is strong to those where it is weak and of water in the opposite direction, and equilibrium is not attained until the strength of the solution is uniform. This process is called diffusion. In liquids it is an exceedingly slow process. Thus, if the tube containing the copper sulphate solution were a metre long and the lower half were filled with the solution, the upper half with pure water, it would take considerably more than ten years before the mixture became approximately uniform; if the height of the tube were a centimetre, it would take about ten hours, the time required being proportional to the square of the length of the tube.

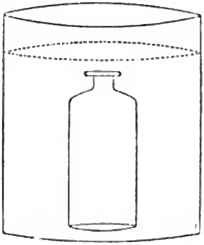

FIG. 144.

The first systematic experiments on diffusion were made by Graham in 1851. The method he used was to take a wide-necked bottle, such as is shown in Fig. 144, and fill it to within a short distance of the top with the salt solution to be examined; the bottle was then carefully filled up with pure water pressed from a sponge on to a disc of cork floating on the top of the solution; the bottle was placed in a larger vessel filled with pure water to about an inch above the top of the bottle. This was left undisturbed for several days, and then the amount of salt which had escaped from the bottle into the outer vessel was determined. Graham was in this way able to show that solutions of the same strength of different substances diffused with different velocities; that solutions of the same salt of different strengths diffused with velocities proportional to the strength; that the rate of diffusion increased with the temperature, and that the proportion of two salts in a mixture was altered by diffusion, and that in some cases a decomposition or separation of the constituents of complicated salts, such as bisulphate of potash and potash alum, could be brought about by diffusion. Though Graham's experiments proved many important and interesting properties of diffusion, they did not lead to sufficiently definite laws to enable us to calculate the state of a mixture at any future time from its state at the present time. This step was made by Fick, who, guided by Fourier's law of the conduction of heat—the diffusion of temperature—enunciated in 1855 the law of diffusion, which has been abundantly verified by subsequent experiments. Fick's law may be stated as follows: Imagine a mixture of salt and water arranged so that layers of equal density are horizontal. Let the state of the mixture be such that in the layer at a height x above a fixed plane there are n grammes of salt per cubic centimetre; then across unit area of this plane $R\frac{dn}{dx}$ grammes of salt will pass in unit time from the side on which the solution is stronger to that on which it is weaker. R is called the diffusivity of the substance; it depends on the nature of the salt and the solvent, on the temperature, and to a slight extent on the strength of the solution. This law is analogous to Fourier's law of the conduction of heat, and the same mathematical methods which give the solution of the thermal problems can be applied to determine the distribution of salt through the liquid. The curves in Figs. 145 and 146 represent the solution of two important problems. The first represents the diffusion of salt from a saturated solution into a vertical column of water, the surface of separation being initially the plane $x = o$. The ordinates represent the amount of salt in the solution at a distance from the original surface of separation represented by the abscissæ. The times which have elapsed since the commencement of diffusion are proportional to the squares of the numbers

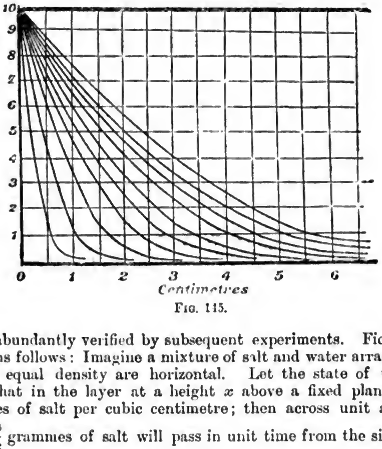

FIG. 145.

on the curve; thus, if the first curve represents the state of things after time T, the second represents it after a time 2^2T, the third after a time 3^2T, and so on; for the same ordinate the abscissa on curve 2 is twice that on curve 1, on curve 3 three times that on curve 1, and so on; thus the time required for diffusion through a given length is proportional to the square of the length. The curves are copied from Lord Kelvin's *Collected Papers*, vol. iii. p. 432: for copper sulphate through water T = 25,700 seconds, for sugar through water T = 17,100, and for sodium chloride through water T = 5390. The second figure, Fig. 146, represents the diffusion when we have initially a thin layer of salt solution at the bottom of a vertical vessel, the rest of the vessel being filled with pure water; the ordinates represent the amount of salt at a distance from the bottom of the vessel represented by the abscissæ. The times which have elapsed since the commencement are proportional to the squares of the numbers on the curves.

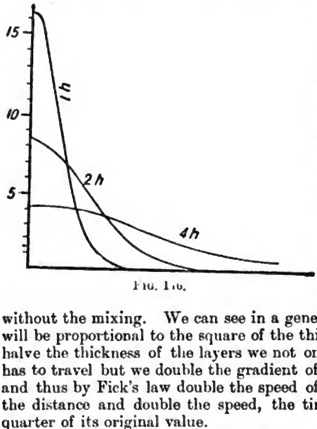

Fig. 146.

By stirring up a solution of a salt with pure water we bring thin layers of the solvent and of the salt near together; as the time required for diffusing through a given distance varies as the square of the distance, the time required for the salt and water to become a uniform mixture is greatly diminished by drawing out the liquid into these thin layers by stirring, and as much diffusion will take much in a few seconds as would take place in as many hours without the mixing. We can see in a general way why the time required will be proportional to the square of the thickness of the layers; for if we halve the thickness of the layers we not only halve the distance the salt has to travel but we double the gradient of the strength of the solution, and thus by Fick's law double the speed of diffusion; thus, as we halve the distance and double the speed, the time required is reduced to one quarter of its original value.

Methods of Determining the Coefficient of Diffusion.—If we know the original distribution of the salt through the water and the value of R, we can, by Fourier's mathematical methods, calculate the distribution of salt after any interval T; conversely, if we know the distribution after this interval, we can use the Fourier result to determine the value of R. Thus, if we have any means of measuring the amount of salt in the different parts of the solution at successive intervals, we can deduce the value of R. It is not advisable to withdraw a sample from the solution and then determine its composition, as the withdrawal of the sample might produce currents in the liquids whose effects might far outweigh any due to pure diffusion; it is, therefore, necessary to sample the composition of the solution when *in situ*, and this has been done by measuring some physical property of the solution which varies in a known way with the strength of the solution. In Lord Kelvin's method the specific gravity is the property investigated: the lower half of a vertical vessel is filled with a solution, the upper half with pure water. Glass beads of different densities are placed in the solution; at first they float at the junction of the solution and the water, but as diffusion goes on they separate out, the heavier ones sink and the lighter ones rise. By noting the position of the beads of known density we can get the distribution of salt in the solution, and thence deduce the value of R. The objection to the method is that air bubbles are apt to form on the beads when salt will crystallise out on them, and thus alter their buoyancy. In the case of sugar solutions the strength of the different layers can be determined by the rotation of the plane of polarisation. H. F. Weber verified Fick's law in the case of zinc sulphate solution by measuring the electromotive force between two amalgamated zinc plates; he had previously determined how the electromotive force depends on the strength of the solutions in contact with the plates. The diffusion of different salts was compared by Long (*Wied. Ann.* 9, p. 613) by the method shown in Fig. 147. A stream of pure water flows through the bent tube, a wide tube fastened on to the bent tube establishes communication with the solution in the beaker; after the water has flowed through the bent tube for some time the amount of salt it carries over in a given time becomes constant. As the water in the tube is continually being renewed, while the strength of the solution in the beaker may be regarded as constant, since in the experiments only a very small fraction of the salt is carried over, the gradient of concentration in the neck will be proportional to the strength of the solution; so that the amount of salt carried off by the stream of water in unit time is proportional to the product of the diffusivity and the strength of the solution. By measuring the amount of salt carried over by the stream in unit time the diffusivities of different salts can be compared. As a result of these experiments it has been found that as a general rule the higher the electrical conductivity of a solution of a salt the more rapidly does the salt diffuse. The relative values of the diffusivity for some of the commoner salts and acids are given in the table on p. 186. The solutions contain the same number of gramme equivalents per litre, and the numbers in the table are proportional to the number of molecules of the salt which cross unit surface in unit time under the same gradient of strength of solution.

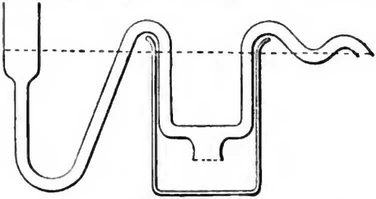

FIG. 147.

Substance.		Substance.	
KCl	803	KI	823
NH_4Cl	689	NaI	672
NaCl	600	NH_4NO_3	680
LiCl	541	KNO_3	607
KCy	767	$NaNO_3$	524
$BaCl_2$	450	$LiNO_3$	512
$SnCl_2$	432	BaN_2O_6	656
$CaCl_2$	429	SrN_2O_6	552
$MgCl_2$	392	$(NH_4)_2SO_4$	724
$COCl_2$	306	Na_2SO_4	678
$NiCl_2$	304	$MgSO_4$	348
KBr	811	$ZnSO_4$	332
NH_4Br	629	$CuSO_4$	316
NaBr	509	$MnSO_4$	298

These numbers show that as a general rule the salts which diffuse the most rapidly are those whose solutions have the highest electrical conductivity. The absolute values of the diffusivity for a large number of substances have been determined by Schuhmeister (*Wien. Akad.* 79, p. 603) and Scheffer (*Chem. Ber.* xv. p. 788, xvi. p. 1903). The largest value of the diffusivity found by Scheffer was for nitric acid; the diffusivity varied with the concentration and with the temperature; for very dilute solutions at 90° C. it was 2×10^{-5} (cm.)^2/sec.—*i.e.*, if the strength of solution varied by one per cent. in 1 cm. the amount of acid crossing unit area in one second would be about one five-millionths of the acid in 1 c.c. of the solution. For solutions of NaCl the diffusivity was only about one half of this value. Graham found that the velocity of diffusion of NaCl through gelatine was about the same as through water.

Diffusion through Membranes. Osmosis.—Graham was led by his experiments on diffusion to divide substances into two classes—crystalloid and colloid. The crystalloids, which include mineral acids and salts, and which as a rule can be obtained in definite crystalline forms, diffuse much more rapidly than the substances called by Graham colloids, such as the gums, albumen, starch, glass, which are amorphous and show no signs of crystallisation. The crystalloids when dissolved in water change in a marked degree its properties: for example, they diminish the vapour pressure, lower the freezing- and raise the boiling-point. Colloidal substances, when dissolved in water, hardly produce any effects of this kind, in fact, many colloidal solutions seem to be little more than mechanical mixtures, the colloid in a very finely divided state being suspended in the fluid. The properties of solutions of this class are very interesting; the particles move in the electric field, in some cases as if they were positively, in others as if they were negatively, charged. The addition of a trace of acid or alkali is often sufficient to produce precipitation. The reader will find an account of the properties of these solutions in papers by Picton and Linder (*Journal of Chemical Society*, vol. 70, p. 568, 1897; vol. 61, p. 148, 1892); Stoeckl and Vanino (*Zeitschrift f. Phys. Chem.*, vol. 30, p. 98, 1899); Hardy (*Proceedings of Royal Society*, 66, p. 110; *Journal of Physiology*, 24, p. 288). Colloidal substances when mixed with not too much water form jellies; the structure of these jellies is sometimes on a sufficiently coarse scale to be visible under the microscope (*see* Hardy, *Proceedings Royal Society*, 66, p. 95, 1900), and apparently consists of a more or less solid framework through which the liquid is dispersed. Through many of these jellies crystalloids are able to diffuse with a velocity approaching that through pure water; the colloids, on the other hand, are stopped by such jellies. Graham founded on this a method for the separation of crystalloids and colloids, called dialysis. In this method a film of a colloidal substance, such as parchment paper (paper treated with sulphuric acid) or a piece of bladder is fastened round the end of a glass tube, the lower end of the tube dipping in water which is frequently changed, and the solution of crystalloids and colloids is put in the tube above the parchment paper. The crystalloids diffuse through into the water, and the colloids remain behind; if time be given and the water into which the crystalloids diffuse be kept fresh, the crystalloids can be entirely separated from the colloids.

The passage of liquids through films of this kind is called osmosis. The first example of it seems to have been observed by the Abbé Nollet, in 1748, who found that when a bladder full of alcohol was immersed in water, the water entered the bladder more rapidly than the alcohol escaped, so that the bladder swelled out and almost burst. If, on the other hand, a bladder containing water was placed in alcohol the bladder shrank.

The motion of fluids through these membranes can be observed with very simple apparatus: all that is necessary is to attach a piece of parchment-paper firmly on the end of a glass tube, the upper portion of which is drawn out into a fine capillary tube. If this tube is filled with a solution of sugar and immersed in pure water, the top of the liquid in the capillary part of the tube moves upwards with sensible velocity, showing the entrance of water through the parchment-paper. Graham regarded this transport of water through the membrane as due to this colloidal substance being able to hold more water in combination when in contact with pure water than when in contact with a salt solution; thus, when the hydration of the membrane corresponding to the side next the water extends to the side next the solution, the membrane cannot hold all the water in combination, and some of it is given up; in this way water is transported from one side of the membrane to the other.

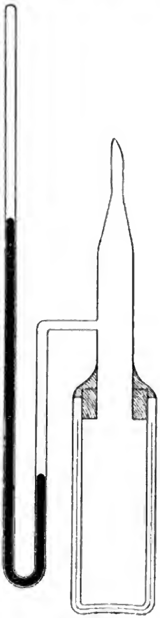

FIG. 118.

Membranes of parchment-paper or bladder are permeable by crystalloids as well as by water. There are other membranes, however, which, while permeable to water are impermeable to a large number of salts; these membranes are called semi-permeable membranes. One of these, which has been extensively used, is the gelatinous precipitate of ferrocyanide of copper, which is produced when copper sulphate and potassium ferrocyanide come into contact. This precipitate is mechanically exceedingly weak, but Pfeffer made serviceable membranes by precipitating it in the pores of a porous pot. If such a pot is filled with a very dilute solution of copper sulphate and immersed in one of ferrocyanide of potassium the two solutions will diffuse into the walls of the pot, and where they meet the gelatinous precipitate of ferrocyanide of copper will be formed; in this way a continuous membrane may be obtained. For details as to the precautions which must be taken in the preparation of these membranes the reader is referred to a paper by Adie (*Proceedings of Chemical Society*, lix. p. 344). If a membrane of this kind be deposited in a porous pot fitted with a pressure gauge, as in Fig. 148, and the pot be filled with a dilute solution of a salt and immersed in pure water, water will flow into the pot and compress the air in the gauge, the pressure in the pot increasing until a definite pressure is reached depending on the strength of the solution. When this pressure is reached there is equilibrium, and there is no further increase in the volume of water in the pot.

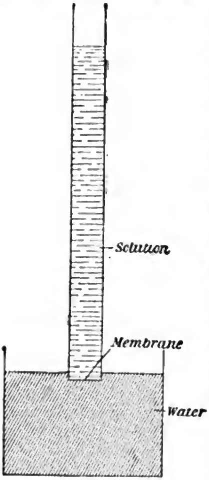

Fig. 149.

Osmotic Pressure.—Thus the flow of water through the membrane into the stronger solution can be prevented by applying to the solution a definite pressure; this pressure is called the osmotic pressure of the solution. It is a quantity of fundamental importance in considering the properties of the solution, as many of these properties, such as the diminution in the vapour pressure, and the lowering of the freezing-point, are determinate as soon as the osmotic pressure is known.

The work done when a volume v of water passes across a semi permeable membrane from pure water into a solution where the osmotic pressure is P is equal to Pv. For, let the solution be enclosed in a vertical tube closed at the bottom by a semi-permeable membrane, then when there is equilibrium the solution is at such a height in the tube that the pressure at the membrane due to the head of the solution is equal to the osmotic pressure. When the system is in equilibrium we know by Mechanics that the total work done during any small alteration of the system must be zero. Let this alteration consist in a volume v of water going through the semi-permeable membrane. This will raise the level of the solution, and the work done against gravity is the same as if a volume v of the solution were raised from the level of the membrane to that of the top of the liquid in the tube. Thus the work done against gravity is $vg\rho h$, where h is the height of the solution in the tube and ρ the density of the solution; since the pressure due to the head of solution is equal to the osmotic pressure, $g\rho h = P$. Hence the work done against gravity by this alteration is Pv, and since the total work done must be zero, the work done *on* the liquid when it crosses the membrane must be P*v*.

The values of the osmotic pressures for different solutions was first determined by Pfeffer,* who found the very remarkable result that for weak solutions which do not conduct electricity the osmotic pressure is equal to the gaseous pressure which would be exerted by the molecules of the salt if these were in the gaseous state and occupying a volume equal to that of the solvent in which the salt is dissolved. Thus, if 1 gramme equivalent of the salt were dissolved in a litre of water the osmotic pressure would be about 22 atmospheres, which is the pressure exerted by 2 grammes of hydrogen occupying a litre. Pfeffer's experiments showed that approximately, at any rate, the osmotic pressure was, like the pressure of a gas, proportional to the absolute temperature. If the cell is placed in another solution instead of pure water, water will tend to run into the cell if the osmotic pressure of the solution in the cell is greater than that of the solution in which it is immersed, while if the osmotic pressure in the cell is less than that outside the volume of water in the cell will decrease; if the osmotic pressure is the same inside and outside there will be no change in the volume of the water inside the cell. Solutions which have the same osmotic pressure are called isotonic solutions. A convenient method of finding the strengths of solutions of different salts which are isotonic was invented by De Vries.† He showed that the membrane lining the cell-wall of the leaves of some plants, such as *Tradescantia discolor*, *Curcuma rubricaulis*, and *Begonia manicata*, is a semi-permeable membrane, being permeable to water but not to salts, or at any rate not to many salts. The contents of the cells contain salts, and so have a definite osmotic pressure. If these cells are placed in a solution having a greater osmotic pressure than their own, water will run from the cells into the solution, the cells will shrink and will present the appearance shown in Fig. 150 *b*. Fig. 150 *a* shows the appearance of the cells when surrounded by water; the weakest solution which produces a detachment of the cell will be approximately isotonic with the contents of the cell. In this way a series

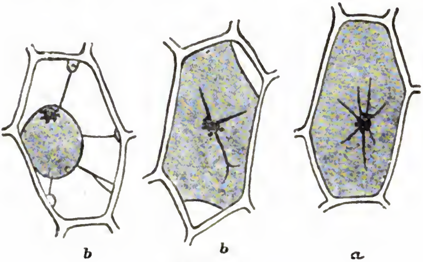

FIG. 150.

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* Pfeffer, *Osmotische Untersuchungen*, Leipzig, 1877.

† De Vries, *Zeit. f. Physik. Chemie*, ii. p 415

of solutions can be prepared which are isotonic with each other. De Vries found that for non-electrolytes isotonic solutions contained in each unit of volume a weight of the salt proportional to the molecular weight; in other words, that isotonic solutions of non-electrolytes contain the same number of molecules of the salt. This is another instance of the analogy between osmotic pressure and gaseous pressure, for it is exactly analogous to Avogadro's law, that when the gaseous pressures are the same all gases at the same temperature contain the same number of molecules per unit volume. Although the direct measurements on osmotic pressure hitherto made may seem a somewhat slight base for the establishment of such an important conception, an immense amount of experimental work has been done in the investigation of such phenomena as the lowering of the vapour pressure, the raising of the boiling- and the lowering of the freezing-point produced by the solution of salts in water. The conception of osmotic pressure enables us to calculate the magnitude of these effects from the strength of the solution; the agreement between the values thus calculated and the values observed is so close as to furnish strong evidence of the truth of this conception.

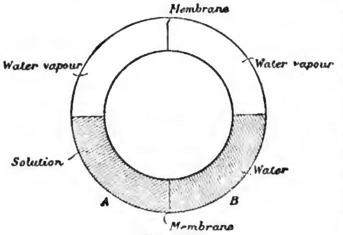

FIG. 151.

Vapour Pressure of a Solution.—The change in the vapour pressure due to the presence of salt in the solution can be calculated by the following method due to Van t' Hoff: Suppose the salt solution *A*, Fig. 151, is divided from the pure water *B* by a semi-permeable membrane—*i.e.*, one which is permeable by water and not by the salt; transfer a small quantity of water whose volume is v from A to B by moving the membrane from right to left. If Π is the osmotic pressure of the solution the work required to effect this transference is Πv; now let a volume v of water evaporate from B and pass as vapour through the membrane into the chamber A and there condense. If V is the volume of the water vapour, δp the excess of the vapour pressure of the water over B above that over A, the work *done* in this process is $\delta p\mathrm{V}$. The process is clearly a reversible one, and hence by the Second Law of Thermodynamics, since the temperatures of the two chambers are the same, there can be no loss or gain of mechanical work. Thus, since the work spent in one part of the cycle must be equal to that gained in the other, we have

$$\Pi v = \delta p\mathrm{V}$$

Suppose p is the vapour pressure over the water, let V′ be the volume occupied at atmospheric pressure Π_0 by the quantity of water vapour which at the pressure p occupies the volume V; then by Boyle's Law,

$$\Pi_0\mathrm{V}' = p\mathrm{V}$$

so that

$$\frac{\delta p}{p} = \frac{\Pi}{\Pi_0} \frac{v}{V'}$$

but for water vapour $v/V' = 1/1200$, hence

$$\frac{\delta p}{p} = \frac{\Pi}{\Pi_0} \frac{1}{1200}$$

The osmotic pressure in a solution of 1 gramme equivalent per litre of a salt which does not dissociate when dissolved is about 22 atmospheres; thus for such a solution

$$\frac{\delta p}{p} = \frac{22}{1200}$$

or the vapour pressure over the solution is nearly 2 per cent. less than over pure water.

If the surface of the solution is subjected to a pressure equal to the osmotic pressure the vapour pressure over the solution will increase and will be equal to the pressure over pure water. For let Fig. 152 represent a vessel divided by a diaphragm permeable only by water and by water vapour, and let the salt solution in *A* be subject to a pressure equal to the osmotic pressure. Under this pressure the liquids will be in equilibrium, and there will be no flow of water across the diaphragm. If the vapour pressure of the water is greater than that of the salt solution, then water vapour from *B* will go across the diaphragm and will condense on *A*; this will make the solution in *A* weaker and reduce the osmotic pressure. Since the external pressure on *A* is now greater than its osmotic pressure, water will flow from *A* to *B* across the diaphragm; thus there would be a continual circulation of water round the system, which would never be in equilibrium. As this is inadmissible, we conclude that the vapour pressure of the water is not greater than that of the solution; similarly if it were less we could show that there would be a continual circulation in the opposite direction; in this way we can show that the vapour pressure of the solution when exposed to the osmotic pressure is equal to that of pure water. This is an example of the theorem proved in J. J. Thomson's *Applications of Dynamics to Physics and Chemistry*, p. 171 (see also Poynting, *Phil. Mag.*, xii. p. 39), that if a pressure of n atmospheres be applied to the surface of a liquid the vapour pressure of the liquid, p, is increased by δp, where

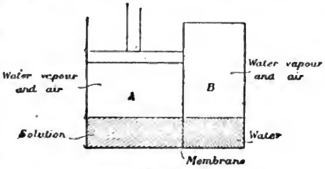

FIG. 152.

$$\frac{\delta p}{p} = n \frac{\text{density of the vapour at atmospheric pressure}}{\text{density of the liquid}}$$

Raising of the Boiling-point of Solutions.—The determination of the vapour pressure is attended with considerable difficulty, and it is much easier to measure the effect of salt on the boiling- or freezing-point of the solution.

Let A and B be vessels containing respectively salt solution and pure water, separated by a semi-permeable membrane, and let the temperatures of the vessels be such that the vapour pressure over the solution is the same as that over pure water. Let θ be the absolute temperature of the water, $\theta+\delta\theta$ that of the solution. Now suppose a volume v of water flows from B to A across the diaphragm; if Π is the osmotic pressure of the solution, mechanical work Πv will be done in this operation. Let this quantity of water be evaporated from A and pass through the walls of the diaphragm and condense in B. As the vapour pressures are the same in the two cases, no mechanical work is gained or spent in this operation. The system is now in its original state, and the operation is evidently a reversible one, so that we can apply the Second Law of Thermodynamics. Now by that law we have

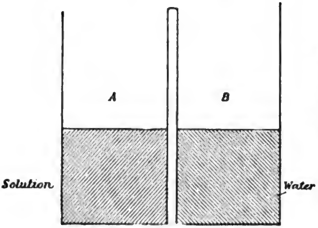

FIG. 153.

$$\frac{\text{Heat taken from the boiler}}{\text{Absolute temperature of boiler}}=\frac{\text{Heat given up in the refrigerator}}{\text{Absolute temperature of refrigerator}}$$

$$=\frac{\text{Mechanical work done by the engine}}{\text{Difference of the temperatures of boiler and refrigerator.}}$$

In our case the mechanical work done is Πv. The heat given up in the refrigerator is the heat given out when a volume v of water condenses from steam at a temperature θ; if λ is the heat given out when unit mass of steam condenses and σ the density of the liquid, the heat given out in the refrigerator is $\lambda\sigma v$; hence by the Second Law we have

$$\frac{\lambda\sigma v}{\theta}=\frac{\Pi v}{\delta\theta} \text{ or } \frac{\delta\theta}{\theta}=\frac{\Pi^{*}}{\lambda\sigma}$$

Let us apply this to find the change in the boiling-point produced by dissolving 1 gramme equivalent of a salt in a litre of water; here Π is 22 atmospheres, or in C.G.S. units 22×10^6. λ is the latent heat of steam in mechanical units—*i.e*, $536\times4{\cdot}2\times10^7$, σ is unity, and $\theta=373$;

hence $$\delta\theta=\frac{373\times22\times10^6}{536\times4{\cdot}2\times10^7}={\cdot}37 \text{ of a degree.}$$

The experiments of Raoult and others on the raising of the boiling-point of solutions of organic salts which do not dissociate have shown

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[*] The heat given out or taken in by the volume of water when going from one chamber to the other is negligible in comparison with that required to vaporise the water

that the amount of the rise in the boiling-point is almost exactly ·37 of a degree for each gramme equivalent per litre, a result which is strong confirmation of the truth of the theory of osmotic pressure.

Lowering of the Freezing-point of Solutions.—A similar investigation enables us to calculate the depression of the freezing-point due to the addition of salt. Let A, B (Fig. 154) represent two vessels separated by a semi-permeable membrane, A containing the salt solution at its freezing-point and B pure water at its freezing-point. Let a volume v of water pass across the semi-permeable membrane from B to A; if Π is the osmotic pressure of the solution, mechanical work Πv will be gained by this process. Let this quantity of water be frozen in A, the ice produced taken from A placed in B, and there melted. The system has now returned to its original condition, and the process is plainly reversible; hence we can apply the Second Law of Thermodynamics. If θ is the absolute temperature of the freezing-point of pure water, $\theta - \delta\theta$ that of the freezing-point of the solution, if λ is the latent heat of water, and σ its density; the heat taken from the hot chamber B at the temperature θ is $\lambda\sigma v$; hence by the Second Law we have

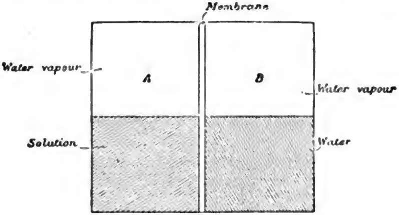

FIG. 154.

$$\frac{\lambda\sigma v}{\theta} = \frac{\Pi v}{\delta\theta} \text{ or } \frac{\delta\theta}{\theta} = \frac{\Pi}{\lambda\sigma}$$

Thus in the case of water for which $\theta = 273$, $\lambda = 80 \times 4{\cdot}2 \times 10^7$, $\sigma = 1$ and when the strength of the solution is 1 gramme equivalent per litre,

$$\Pi = 22 \times 10^6\text{; hence } \delta\theta = 1{\cdot}79^\circ.$$

This has been verified by Raoult in the case of solutions of organic salts and acids. The result of the comparison of theory with experiment for a variety of solvents is shown in the following table:

Solvent	Lowering of freezing point for organic salts, 1 gramme molecule dissolved in a litre	
	Observed	Calculated
Acetic acid	3·9	3·88
Formic acid	2·8	2·8
Benzene	4·9	5·1
Nitro-benzene	7·05	6·9
Ethylene-dibromide	11·7	11·9

Dissociation of Electrolytes.—The preceding theory gives a satisfactory account of the effect upon the boiling- and freezing-points produced by organic salts and acids when the osmotic pressure is calculated on the assumption that it is equal to the gaseous pressure which would be produced by the same weight of the salt if it were gasified and confined in a volume equal to that of the solvent. When, however, mineral salts or acids are dissolved in water, the effect on the boiling- and freezing-points produced by *n* gramme equivalents per litre is greater than that produced by the same number of gramme equivalents of an organic salt, although if the osmotic pressure were given by the same rule, the effects on the freezing- and boiling-points ought to be the same in the two cases. The osmotic pressure then in a solution of a mineral salt or acid is greater than in one of equivalent strength (*i.e.*, one for which *n* is the same) of an organic salt or acid; this has been verified by direct measurement of the osmotic pressure by the methods of Pfeffer and De Vries. This increase in the osmotic pressure is explained by Arrhenius as being due to a partial dissociation of the molecules of the salts into their constituents; thus some of the molecules of NaCl are supposed to split up into separate atoms of Na and Cl. Since by this dissociation the number of individual particles in unit volume is increased, the osmotic pressure, if it follows the law of gaseous pressure, will also be increased. According to Arrhenius, the atoms of Na and Cl into which the molecule of the salt is split are charged respectively with positive and negative electricity, which, as they move under electric forces, will make the solution a conductor of electricity. In this way he accounts for the fact that those solutions in which the osmotic pressure is abnormally large are conductors of electricity, and that, as a rule, the greater the conductivity the greater the excess of the osmotic pressure. This view, of which an account will be given in the volume on Electricity, has been very successful in connecting the various properties of solutions.

Though the osmotic pressure plays such an important part in the theory of solution, there is no generally accepted view of the way in which the salt produces this pressure. One view is that the salt exists in the interstices between the molecules of the solvent in the state corresponding to a perfect gas. If the volume of these interstices bore a constant proportion to the volume of the solvent, then, whatever this ratio may be, we should get the ordinary relation between the quantity of salt and the osmotic pressure to which it gives rise. For, suppose p is the pressure of the gaseous salt, v the volume of the interstices, V the volume of the solvent; then if a semi-permeable membrane be pushed so that a volume δV of water passes through it, and Π is the osmotic pressure, then the work done is $\Pi\delta V$; but if δv is the diminution in the volume of the interstices, the work done is $p\delta v$; hence

$$\Pi\delta V = p\delta v$$

But if the volume occupied by the interstices bears a constant ratio to that of the solvent

$$\frac{\delta V}{V} = \frac{\delta v}{v}$$

where V is the volume of the solvent; hence

$$\Pi V = pv \quad \text{or} \quad \Pi = \frac{pv}{V};$$

that is, the osmotic pressure is the same as if the gaseous salts occupied the whole volume of the solvent.

Another view (*see* Poynting, *Phil. Mag.* 42, p. 289) is that the phenomenon known as osmotic pressure arises from the molecules of salt clinging to the molecules of the water, and so diminishing the mobility and therefore the rate of diffusion of the latter. Thus, suppose we have pure water and a salt solution separated by a semi-permeable membrane, since the water molecules in the solution are clogged by the salt they will not be able to pass across the membrane as quickly as those from the pure water, and there will be a flow of water across the membrane from the pure water to the solution. Poynting shows that the mobility of the molecules of a liquid is increased by pressure, so that by applying a proper pressure to the solution we may make the mobility of the molecules of water in it the same as those of the pure water, and in this case there will be no flow across the membrane; the pressure required is the osmotic pressure. Poynting shows that this view will explain the properties of inorganic salts if we suppose that each molecule of salt can completely destroy the mobility of one molecule of water.

CHAPTER XVII.

DIFFUSION OF GASES.

CONTENTS.—Co-efficient of Diffusion—Diffusion of Vapours—Explanation of Diffusion on Kinetic Theory of Gases—Effects of a Perforated Diaphragm—Passage of Gases through Porous Bodies—Thermal Effusion—Atmolysis—Passage of Gases through Indiarubber, Liquids, Hot Metals—Diffusion of Metals through Metal.

IF a mixture of two gases A and B is confined in a vessel the gases will mix and each will ultimately be uniformly diffused through the vessel as if the other were not present. If they are not uniformly mixed to begin with, there will be a flow of the gas A from the places where the density of A is great to those where it is small. The law of this diffusion is analogous to that of the conduction of heat or to the diffusion of liquids and may be expressed mathematically as follows: Suppose the two gases are arranged so that the layers of equal density are horizontal planes, and let ρ be the density of A at a height x above a fixed horizontal plane; then in unit time the mass of A which passes downward through unit area of a horizontal plane at a height x is proportional to the gradient of ρ and is equal to $K\frac{d\rho}{dx}$ where K is the interdiffusity of the gases A and B. The value of K has been measured by Loschmidt[*] and Obermayer[†] for a considerable number of pairs of gases. The method employed by these observers was to take a long vertical cylinder separated into two parts by a disc in the middle. The lower half of the cylinder was filled with the heavier gas, the upper half with the lighter. The disc was then removed with great care so as not to set up air currents, and the gases were then allowed to diffuse into each other; after the lapse of a certain time the disc was replaced and the amount of the heavier gas in the upper half of the cylinder determined. From this the value of K was determined on the assumption (which is probably only approximately true) that the value of K does not change when the proportions of the two gases are altered. Waitz[‡] used a different method to determine the coefficient of interdiffusion of air and carbonic acid; beginning with the carbonic acid below the air he measured by means of Jamin's interference refractometer the refractive index of various layers after the lapse of definite intervals of time; from the refractive index he could calculate the proportion of air and carbonic acid gas, and was thus able to follow the course of the diffusion. He found that the coefficient of diffusion depended to some extent on the proportion between the two gases, the values of K at atmospheric pressure at 0° C. varying between ·1288 and ·1366 cm.^{2}/sec. The values found by Loschmidt and v. Obermayer are given in the following table. They are for 76 cm. pressure and 0° C:

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[*] Loschmidt, *Wien. Berichte*, 61, p. 367, 1870, 62, p. 468, 1870.

[†] Obermayer, *Wien. Berichte*, 81, p. 162, 1880.

[‡] Waitz, *Wiedemann's Annalen*, 17, p. 201, 1882.

Gases.	LOSCHMIDT. K cm.^{2}/sec.	VON OBERMAYER. K cm.^{2}/sec.
$CO_2 - N_2O$	·09831	·09166
$CO_2 - CO$	·14055	·13142
$CO_2 - O_2$	·14095	·13569
CO_2 – Air	·14231	·13433
$CO_2 - CH_4$	·15856	·14650
$CO_2 - H_2$	·55585	·53409
$CO_2 - C_2H_4$	—	·10061
$CO - O_2$	·18022	·18717
$CO - H_2$	·64223	·64884
$CO - C_2H_4$	—	·11639
$SO_2 - H_2$	·48278	—
$O_2 - H_2$	·72167	·66550
$O_2 - N_2$	—	·17875
O_2 – Air	—	·17778
H_2 – Air	—	·63405
$H_2 - CH_4$	—	·62544
$H_2 - N_2O$	—	·53473
$H_2 - C_2H_6$	—	·45933
$H_2 - C_2H_4$	—	·48627

We may, perhaps, gain some idea of the rapidity of diffusion by saying that the rate of equalisation in composition of a mixture of hydrogen and air is about half that of the equalisation of temperature in copper.

As an example of the rate at which diffusion goes on we may quote the result of an experiment by Graham on the diffusion of CO_2 into air. Carbonic acid was poured into a vertical cylinder 57 cm. high until it filled one-tenth of the cylinder. The upper nine-tenths of the vessel was filled with air and the gases were left to diffuse. They were found to be very approximately uniformly distributed throughout the cylinder after the lapse of about two hours. As the time taken to reach a state of approximately uniform distribution is proportional to the square of the length of the cylinder, if the cylinder were only one centimetre long approximately uniform distribution would be attained after the lapse of about two seconds.

The interdiffusity is inversely proportional to the pressure of the mixed gas; it increases with the temperature. According to the experiments of Loschmidt and v. Obermayer it is proportional to θ^n where θ is the absolute temperature and n a quantity which for different pairs of gases varies between 1·75 and 2.

Diffusion of Vapours.—The case when one of the diffusing gases is the vapour of a liquid is of special importance, as it is on the rate of diffusion that the rate of evaporation depends. The methods which have been employed to measure the rate of diffusion of the vapour of a liquid consist essentially in having some of the liquid at the bottom of a cylindrical tube and directing a blast of vapour-free gas across the mouth of the tube. When the blast has been blowing for some time a uniform gradient of the density of the vapour is established in the tube, the value of this is δ/l where δ is the maximum vapour pressure of the liquid at the temperature of the experiment and l the distance of the surface of the liquid from the mouth of the tube. The mass of vapour which in unit time flows out of the tube—(*i.e.*, the amount of the liquid which evaporates in unit time and which can therefore be easily measured), is $K\delta/l$ where K is the diffusivity of the vapour into the gas; as δ is known we can readily determine K by this method. A few of the results of experiments made by Stefan* and Winkelmann† are given in the following table:

VALUE OF K IN cm.²/sec. AT 0° C. AND 760 mm. PRESSURE.

	Hydrogen.	Air.	Carbonic acid.
Water-vapour	·687	·198	·131
Ether	·296	·0775	·0552
Carbon-bisulphide	·369	·0883	·0629
Benzol	·294	·0751	·0527
Methyl-alcohol	·5001	·1325	·0880
Ethyl-alcohol	·3806	·0994	·0693

Explanation of Diffusion on the Kinetic Theory of Gases.—The kinetic theory according to which a gas consists of a great number of individual particles called molecules in rapid motion, affords a ready explanation of diffusion. Suppose we have two layers A and B in a mixture of gases and that these layers are separated by a plane C. Let there be more molecules of some gas γ in A than in B, then since the molecules are in motion they will be continually crossing the plane of separation, some going from A to B and some from B to A, but inasmuch as the molecules of γ in A are more numerous than those in B, more will pass from A to B than from B to A. Thus, A will lose and B gain some of the gas γ; this will go on until the quantities of γ in unit volumes of the layers A and B are equal, when as many molecules will pass from A to B as from B to A, and thus the equality, when once established, will not be disturbed by the motion of the molecules. It follows from the kinetic theory of gases (see Boltzmann, *Vorlesungen über Gastheorie*, p. 91) that, if there are n molecules of γ in unit volume of B, $n + \delta n$ in a unit volume of A at a distance δx from that in B, and if x be measured at right angles to the plane separating the layers, then the excess of the number of molecules of γ which go across unit area of C from A to B over those which go from A to B is equal to $\cdot 3502\lambda\bar{c}\frac{dn}{dx}$, where λ is the mean free path of the molecules of γ and $\bar{c}$, their average velocity of translation; the quantity $\lambda\bar{c}$ is evidently proportional to the diffusity.

Now $\bar{c}$ only depends upon the temperature, being proportional to the square root of the absolute temperature, while λ is inversely proportional to the density, and if the density is given it does not, at least if the molecules are regarded as hard elastic spheres, depend upon the temperature. If the pressure is given, then the density will be inversely, and λ therefore directly proportional to the absolute temperature. Thus, on this theory the coefficient of diffusion should vary as $\theta^{\frac{3}{2}}$ where θ is the absolute temperature. The experiments of Loschmidt and von Obermayer seem to show that it varies somewhat more rapidly with the temperature.

Another method of regarding the process of diffusion, which for some purposes is of great utility, is as follows: The diffusion of one gas A through another B when the layers of equal density are at right angles to

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[*] Stefan, *Wien. Akad. Ber.*, 65, p. 323, 1872.

[†] Winkelmann, *Wied. Ann.*, 22, pp. 1 and 152, 1884.

the axis x may be regarded as due to a current of the gas A moving parallel to the axis of x with a certain velocity u through a current of B streaming with the velocity v in the opposite direction. To move a current of one gas through another requires the application of a force to one gas in one direction and an equal force to the other gas in the opposite direction. This force will be proportional (1) to the relative velocity $u+v$ of the two currents, (2) to the number of molecules of A per unit volume, and (3) to that of the molecules of B. Let it then per unit volume of gas be equal to $A_{12}\,\rho_1\rho_2\,(u+v)$, where A_{12} is a quantity depending on the nature of the gases A and B, but not upon their densities nor upon the velocity with which they are streaming through each other; ρ_1 and ρ_2 are respectively the densities of the gases A and B—*i.e.*, their masses per unit volume. Hence, to sustain the motion of the gases a force $A_{12}\,\rho_1\,\rho_2\,(u+v)$ parallel to x must act on each unit of volume of A and an equal force in the opposite direction on each unit volume of B. These forces may arise in two ways; there may be external forces acting on the gases, and there may also be forces arising from variations in the partial pressures due to the two gases. Let X_1, X_2 be the external forces per unit mass acting on the gases A and B respectively, and p_1, p_2 the partial pressures of the gases A and B respectively. Considering the forces acting parallel to x on unit volume of A, the external force is $X_1\rho_1$, and the force due to the variation of the partial pressure is $-dp_1/dx$; hence the total force is equal to $-dp_1/dx+X_1\rho_1$, and as this is the force driving A through B we have

$$-\frac{dp_1}{dx}+X_1\rho_1=A_{12}\rho_1\rho_2(u+v) \qquad (1)$$

similarly,

$$-\frac{dp_2}{dx}+X_2\rho_2=-A_{12}\rho_1\rho_2(u+v) \qquad (2)$$

Let us consider the case when there are no external forces and when the total pressure p_1+p_2 is constant throughout the vessel in which diffusion is taking place. In this case the number of molecules of A which cross unit area in unit time must equal the number of molecules of B which cross the same area in the same time in the opposite direction. Let this number be q; then if n_1 n_2 are respectively the numbers of molecules of A and B per unit volume,

$$q=n_1u=n_2v$$

If m_1, m_2 are the masses of the molecules of A and B respectively

$$\rho_1=m_1n_1 \qquad \rho_2=m_2n_2$$

hence

$$A_{12}\rho_1\rho_2(u+v)=A_{12}m_1m_2(n_1+n_2)q$$

Now n_1+n_2 is proportional to the total pressure, and as this is constant throughout the volume, n_1+n_2 will be constant. Putting $X=0$ in equation (1) and writing N for n_1+n_2, we get

$$q=-\frac{1}{A_{12}m_1m_2N}\frac{dp_1}{dx}$$

Now

$$\frac{p_1}{n_1}=\frac{p_2}{n_2}$$

where n_o is the number of molecules of a gas in unit volume at a standard pressure p_o;

hence
$$q = -\frac{p_o}{Nn_oA_{12}m_1m_2}\frac{dn_1}{dx}$$

Now q is the number of molecules of A passing unit surface in unit time and dn_1/dx is the gradient of the number per unit volume; hence, from the definition of K, the interdiffusity, given on p. 196, we see

$$K = \frac{p_o}{Nn_oA_{12}m_1m_2}$$

or if P is the total pressure

$$K = \frac{1}{m_1m_2PA_{12}}\left(\frac{p_o}{n_o}\right)^2$$

Thus, if A_{12} is constant, K varies inversely as P, and directly as $(p_0/n_0)^2$. Since the pressure of a given number of molecules per unit volume is proportional to the absolute temperature, K, if A_{12} is constant, varies directly as the square of the absolute temperature.

We can determine A_{12} if we know the velocity acquired by one of the gases A when acted upon by a known force. Suppose that the gas A is uniformly distributed, so that $dp_1/dx = 0$, and that when acted upon by a known force it moves through B with a velocity u; suppose, too, that B is very largely in excess and is not acted upon by the force, we have then v very small compared with u, and from equation (1) we have

$$A_{12} = \frac{X}{\rho_2 u}$$

Thus, if we know u, the velocity acquired under a known force X, we can find A_{12}, and hence K, the diffusivity. This result is of great importance in the theory of the diffusion of ions in electrolytes, and Nernst has developed an electrolytic theory of diffusion in fluids on this basis. Another important application of this result is to determine X from measurements of K and u. Thus, to take an example, if the particles of the gas A are charged with electricity and placed in an electric field of known strength, the force X will depend upon the charge; hence, if in this case we measure (as has been done by Townsend) the values of K and u, we can deduce the value of X, and hence the charge carried by the particles of A.

On the Obstruction offered to the Diffusion of Gases by a perforated Diaphragm.—If a perforated diaphragm is placed across a cylinder it does not diminish the diffusion of gases in the cylinder in the ratio of the area of the openings in the diaphragm to the whole area of the diaphragm, but in a much smaller degree, for the effect of the perforation is to make the gradient in the density of the gases in the neighbourhood of the hole greater than it would have been if the diaphragm had been removed, and therefore the flow through the hole greater than through an equal area when there is no diaphragm. Thus, to take a case investigated by Dr. Horace Brown and Mr. Escombe (*Proceedings Royal Society*, vol. 67, p. 124), suppose we have CO_2 in a cylinder, and place across the cylinder a disc wet with a solution of caustic alkali which absorbs the CO_2, so that the density of the CO_2 next the disc is zero. Then if ρ is the density of the CO_2 at the top of the cylinder, the density gradient is ρ/l where l is the distance between the disc and the top of the cylinder, so that the amount of CO_2 absorbed by unit area of the disc will be $k\rho/l$ where k is the diffusivity of CO_2 through itself. Now suppose, instead of a disc extending completely across the cylinder, we have a much smaller disc of radius a, then at the disc the density of the CO_2 will be zero, but it will recover its normal value ρ at a distance from the disc proportional to a; thus the gradient of density in the neighbourhood of the disc will be of the order ρ/a and not ρ/l, and the amount of CO_2 absorbed by the disc will be proportional to $k\ (\rho/a)\ \pi a^2$—*i.e.*. will be proportional to a; so that the absorption of the CO_2 will only diminish as the radius of the disc and not as the area. This was verified by Brown and Escombe, and it has very important applications to the passage of gases through the openings in the leaves of plants.

Passage of Gases through Porous Bodies.—There are three processes by which gas may pass through a solid perforated by a series of holes or canals; the size of the holes or pores determining the method by which the gas escapes. If the plate is thin and the pores are not exceedingly fine, the gas escapes by what is called effusion; this is the process by which water or air escapes from a vessel in which a hole is bored. The rate of escape is given by Torricelli's theorem, so that the velocity with which a gas streams through an aperture into a vacuum is proportional to the square root of the quotient of the pressure of the gas by its density, and thus for different gases under the same pressure the velocity will vary inversely as the square root of the density of the gas. Bunsen founded on this result a method of finding the density of gases. This case, strictly speaking, is not one of diffusion at all, but merely the flow of the gas as a whole through the aperture. If the gas is a mixture of different gases its composition will not be altered when the gas passes through an aperture of this kind.

The second method is the one which occurs when the holes are not too fine, and when the thickness of the plate is large compared with the diameter of the holes. In this case the laws are the same as when a gas flows through long tubes; they depend on the viscosity of the gas, and are discussed in the chapter relating to that property of bodies. No change in the composition of a mixture of gases is produced when the gases are forced through apertures of this kind; this is again a motion of the gas as a whole, and not a true case of diffusion. The third method occurs when the pores are exceedingly fine, such as those found in plates of meerschaum, stucco, or a plate of graphite prepared by squeezing together powdered graphite until it forms a coherent mass. In this case, when we have a mixture of two gases, each finds its way through the plate independently of the other, and the composition of the mixture is in general altered by the passage of the gas through the plate. The laws governing the passage of gases through pores of this kind were investigated by Graham, who found that the volume of the gas (estimated at a standard pressure) passing through a porous plate was directly proportional to the difference of the pressures of the gas on the two sides, and inversely proportional to the square root of the molecular weight of the gas. Thus for the same difference of pressure hydrogen was found to escape through a plate of compressed graphite at four times the rate of oxygen. Thus, if we have mixtures of equal volumes of hydrogen and oxygen and allow them to pass through a porous diaphragm, since the hydrogen gets through at four times the rate of the oxygen, the mixture, after passing through the plate, will be much richer in hydrogen than in oxygen. The rate of diffusion can be measured by an instrument of the following kind (Fig. 155): A porous plate is fastened on the top of a tube which can be used as a barometer tube. A vessel for holding the gas being attached to the upper part of the tube, this and the space above the mercury are exhausted; gas at a definite pressure is then let into the vessel, and the rate at which it passes through the diaphragm into the vacuum over the mercury is measured by the rate of depression of the mercury column.

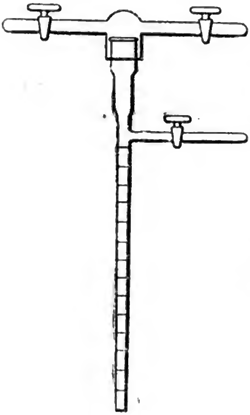

FIG. 155.

The laws of diffusion of gases through fine pores are readily explained by the Kinetic Theory of Gases; for if the pores are so fine that the molecules pass through them without coming into collision with other molecules, the rate at which the molecules pass through will be proportional to the average velocity of translation of the molecules. According to the Kinetic Theory of Gases this average velocity is inversely proportional to the square root of the molecular weight of the gas and directly proportional to the square root of the absolute temperature. Hence at a given temperature the velocity with which the gas streams through the apertures will be inversely proportional to the square root of the molecular weight; this is the result discovered by Graham.

Thermal Effusion.—The same reasoning will explain another phenomenon sometimes called thermal effusion. Suppose we have a vessel divided into two portions by a porous diaphragm; let the pressures in the two portions be equal but their temperatures different, then gas will stream from the cold to the hot part of the vessel through the diaphragm. For since the pressures are equal the densities in the two parts of the vessel are inversely proportional to the absolute temperatures while the velocities are directly proportional to the square roots of the absolute temperatures. Hence the number of molecules passing from the gas through the diaphragm, which is proportional to the product of the density and the velocity, will be inversely proportional to the square root of the absolute temperature; thus more gas will pass from the cold side than from the hot, and there will be a stream of gas from the cold to the hot portion through the diaphragm.

Atmolysis.—The diffusion of gases through porous bodies was applied by Graham to produce the separation of a mixture of gases; this separation was called by him atmolysis, and to effect it he used an instrument of the kind shown in Fig. 156. A long tube made from the stems of clay tobacco-pipes is fixed by means of corks in a glass or metal tube. A glass tube is inserted in one of the end corks, and is connected with an air-pump so that the annular space between the tobacco-pipes and the outer tube can be exhausted. The mixed gases whose constituents have to be separated is made to flow through the clay pipes. Some of the gases escape through the walls and can be pumped away and collected while the rest flows on through the tube. In the gas which passes through the walls of the tube there is a greater proportion of the lighter gas than there was in the mixture originally, while in the gas which flows along the tube there is a greater proportion of the heavier constituent. If the constituents of the mixture differ much in density a considerable separation of the gases may be produced by this arrangement.

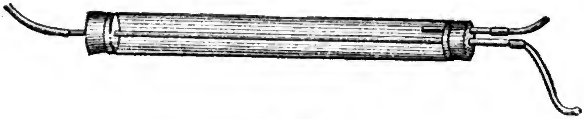

FIG. 156.

Passage of a Gas through India-rubber.—The fact that gases can pass through thin india-rubber was discovered in 1831 by Mitchell, who found that india-rubber toy-balloons collapsed sooner when inflated with carbonic acid than with hydrogen or air, and sooner with hydrogen than air. The subject was investigated by Graham, who gave the following table for the volumes of different gases which pass through india-rubber in the same time:

N_2	1	O_2	2·556
CO	1·13	H_2	5·5
Air	1·149	CO_2	13·585
CH_4	2·148		

The speed with which the gases pass through the rubber increases very rapidly with its temperature. There is no simple relation between these volumes and the densities of the gas as there is in the case of diffusion through a porous plate, and the mechanism by which the gases effect their passage is probably quite different in the two cases. The passage of gases through rubber seems to have many points of resemblance to the passage of liquids through colloidal membranes such as parchment-paper or bladder. The rubber is able to absorb and retain a certain amount of carbonic acid gas, this amount increasing with the pressure of the gas in contact with the surface of the rubber. Thus the layers of rubber next the CO_2 first get saturated with the gas, and this state of saturation gets transmitted from layer to layer; but as on the other side of the sheet of rubber the pressure of the CO_2 is less, the outer layers cannot retain the whole of their CO_2 so that some of the gas gets free.

Passage of a Gas through Liquids.—This is probably analogous to the last case; the gases which are most readily absorbed by the liquid are those which pass through it most rapidly.

Passage of Gases through red-hot Metal.—Deville and Troost found that hydrogen passed readily through red-hot platinum and iron. No gas besides hydrogen is known to pass through platinum. Troost found that oxygen diffused through a red-hot silver tube; quartz is said to be penetrable at high temperatures by the gases from the oxyhydrogen flame.

Diffusion of Metals through Metals.—Daniell showed that mercury diffused through lead, tin, zinc, gold, and silver. Henry proved the diffusion of mercury through lead by a very striking experiment: he took a bent piece of lead and placed the lower part of the shorter arm in contact with mercury; after the lapse of some time he found that the mercury trickled out of the longer arm. He also showed the diffusion of two solid metals through each other by depositing a thin layer of silver on copper; when this was heated the silver disappeared, but on etching away the copper surface silver was found. A remarkable series of experiments on the diffusion of metals through lead, tin and bismuth has been made by Sir W. Roberts-Austen[*]; his results are given in the following table. K is the diffusivity:

Diffusing Metal.	Solvent.	Temperature.	K cm.²/sec.
Gold	Lead	492°	$3{\cdot}47 \times 10^{-5}$
,,	,,	492°	$3{\cdot}55 \times 10^{-5}$
Platinum	,,	492°	$1{\cdot}96 \times 10^{-5}$
,,	,,	492°	$1{\cdot}96 \times 10^{-5}$
Gold	,,	555	$3{\cdot}69 \times 10^{-5}$
,,	Bismuth	555	$5{\cdot}23 \times 10^{-5}$
,,	Tin	555	$5{\cdot}38 \times 10^{-5}$
Silver	,,	555	$4{\cdot}77 \times 10^{-5}$
Lead	,,	555	$3{\cdot}68 \times 10^{-5}$
Gold	Lead	550	$3{\cdot}69 \times 10^{-5}$
Rhodium	,,	550	$3{\cdot}51 \times 10^{-5}$

It will be seen from these results that the rate of diffusion of gold through lead at about 500° is considerably greater than that of sodium chloride through water at 18° C. Sir W. Roberts-Austen has lately shown that there is an appreciable diffusion of gold through solid lead kept at ordinary atmospheric temperatures.

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[*] Roberts-Austen, *Phil. Trans.* A., 1896, p. 393.

CHAPTER XVIII.

VISCOSITY OF LIQUIDS.

CONTENTS.—Definition of Viscosity—Flow of Liquid through Capillary Tube—Flow of Gas through Capillary Tube—Methods of Measurement of Co-efficients of Viscosity—Effect of Temperature and Pressure on Viscosity of Liquids—Viscosity of Solutions and Mixtures—Lubrication—Explanation of Viscosity of Gases on Kinetic Theory—Mean-free Path—Effects of Temperature and Pressure on Viscosity of Gases—Viscosity of Gaseous Mixtures—Resistance to Motion of a Solid through a Viscous Fluid.

A FLUID, whether liquid or gaseous, when not acted on by external forces, moves like a rigid body when in a steady state of motion. When in this state there can be no motion of one part of the liquid relative to another; if such relative motion is produced, say by stirring the liquid, it will die away soon after the stirring ceases. Thus, for example, when a stream of water flows over a fixed horizontal plane, since the top layers of the stream are moving while the bottom layer in contact with the plane is at rest, one part of the stream is moving relatively to the other, but this relative motion can only be maintained by the action of an external force which makes the pressure increase as we go up stream. If this force were withdrawn the whole of the stream would come to rest. The slowly moving liquid near the bottom of the stream acts as a drag on the more rapidly moving liquid near the top, and there are a series of tangential forces acting between the horizontal layers into which we may suppose the stream divided; thus the force acting along a surface such as AB tends to retard the more rapidly moving liquid above it and accelerate the motion of the liquid below it; it thus tends to equalise the motion, and if there were no external forces these tangential stresses would soon reduce the fluid to rest. The property of a liquid whereby it resists the relative motion of its parts is called viscosity. The law of this viscous resistance was formulated by Newton (*Principia*, Lib. II., Sec. 9). It may be stated as follows: Suppose that a stratum of liquid of thickness c is moving horizontally from left to right and that the horizontal velocity, which is nothing at CD, increases uniformly with the height of the liquid, and let the top layer be moving with the velocity V; then the tangential stress which may be supposed to act across each unit of a surface such as AB is proportional to the gradient of the velocity—*i.e.*, to V/c—and tends to stop the relative motion, the tangential stress on the liquid below AB being from left to right, that on the liquid above AB from right to left. The ratio of the stress to the velocity gradient is called the co-efficient of viscosity of the fluid; we shall denote it by the symbol η. The viscosity may be defined in terms of quantities, which may be directly measured as follows: "The viscosity of a substance is measured

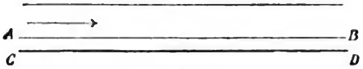

FIG. 157.

by the tangential force on unit area of either of two horizontal planes at unit distance apart, one of which is fixed, while the other moves with the unit of velocity, the space between being filled with the viscous substance" (Maxwell's *Theory of Heat*).

It will be seen that there is a close analogy between the viscous stress and the shearing stress in a strained elastic solid. If a stratum of an elastic solid, such as that in Fig. 157, is strained so that the horizontal displacement at a point P is proportional to the height of P above the plane CD, the tangential stress is equal to $n \times$ (gradient of the displacement) where n is the rigidity of the substance. The viscous stress is thus related to the velocity in exactly the same way as the shearing stress is related to the displacement. This analogy is brought out in the method of regarding viscosity introduced by Poisson and Maxwell. According to this view, a viscous liquid is regarded as able to exert a certain amount of shearing stress, but is continually breaking down under the influence of the stress. We may crudely represent the state of things by a model formed of a mixture of matter in states A and B, of which A can exert shearing stress while B cannot, while under the influence of the stress matter is continually passing from the state A to the state B. If the rate at which the shear disappears from the model is proportional to the shear, say $\lambda\theta$, where θ is the shear, then, when things are in a steady state, the rate at which unit of volume of the substance is losing shear must be equal to the rate at which shear is supplied to it. If ξ is the horizontal displacement of a point at a distance x from the plane of reference, then $\theta = \frac{d\xi}{dx}$. The rate at which shear is supplied to unit volume is $d\theta/dt$ or $\frac{d}{dx}\frac{d\xi}{dt}$; but $d\xi/dt$ is equal to v, the horizontal velocity of the particle, hence the rate at which the shear is supplied is dv/dx. Thus, in the steady state,

$$\frac{dv}{dx} = \lambda\theta.$$

If n is the coefficient of rigidity, the shear θ will give a tangential stress equal to $n\theta$ or

$$\frac{n}{\lambda}\frac{dv}{dx}.$$

If η is the coefficient of viscosity, the viscous tangential stress is equal to

$$\eta\frac{dv}{dx}.$$

Hence, if the viscous stress arises from the rigidity of the substance,

$$\eta = n/\lambda.$$

The quantity $1/\lambda$ is called the time of relaxation of the medium; it measures the time taken by the shear to disappear from the substance when no fresh shear is supplied to it.

This view of the viscosity of liquids is the one that naturally suggests itself when we approach the liquid condition by starting from the solid state; if we approach the liquid condition by starting from the gaseous state we are led (*see* p. 218) to regard viscosity as analogous to diffusion and as arising from the movement of the molecules from one part of the substance to another. This point of view will be considered later.

Flow of a Viscous Fluid through a Cylindrical Capillary Tube.—When the fluid is driven through the tube by a constant difference of pressure it settles down into a steady state of motion such that each particle of the fluid moves parallel to the axis of the tube, provided that the velocity of the fluid through the tube does not exceed a certain value depending on the viscosity of the liquid and the radius of the tube. The relation between the difference of pressure at the beginning and end of the tube and the quantity of liquid flowing through the tube in unit time can be determined as follows:

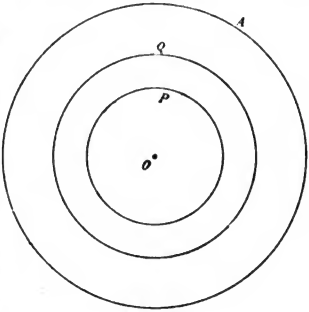

FIG. 158.

Let the cross-section of the tube be a circle of radius OA $= a$, let v be the velocity of the fluid parallel to the axis of the tube at a point P distant r from this axis. Then dv/dr is the gradient of the velocity, and the tangential stress due to the viscosity is $\eta dv/dr$: this stress acts parallel to the axis of the tube. Consider the portion of fluid bounded by two coaxial cylinders through P and Q and by two planes at right angles to the axis of the tube at a distance Δz apart. Let r, $r + \Delta r$ be the radii of the cylinder through P and Q respectively. The tangential stress due to viscosity acting in the direction to diminish v is at P equal to $\eta \frac{dv}{dr}$; the area of the surface of the cylinder through P included between the two planes is $2\pi r \Delta z$, hence the total stress on this surface is

$$2\pi\eta r \frac{dv}{dr} \Delta z$$

Similarly the stress acting on the surface of the cylinder through Q included between the two planes is

$$2\pi\eta \left\{ r\frac{dv}{dr} + \frac{d}{dr}\left(r\frac{dv}{dr}\right)\Delta r \right\} \Delta z$$

and this acts in the direction to increase v; hence the resultant stress tending to increase v is equal to

$$2\pi\eta \frac{d}{dr}\left(r\frac{dv}{dr}\right)\Delta r \Delta z$$

Besides these tangential forces there are the pressures acting over the plane ends of the ring; if Π denote the pressure gradient—*i.e.*, the increase of pressure per unit length in the direction of v, then the effect of the pressures over the ends of the ring is equivalent to a force $2\pi r\Delta r.\Pi\Delta z$ tending to diminish v. Since the motion is steady there is no change in the momentum of the fluid, hence the force tending to diminish v must be equal to that tending to increase it; we thus get

$$2\pi\eta\frac{d}{dr}\left(r\frac{dv}{dr}\right)\Delta r\Delta z = 2\pi r\Pi\Delta r\Delta z$$

or

$$\eta\frac{d}{dr}\left(r\frac{dv}{dr}\right) = r\Pi \tag{1}$$

Now since the liquid is moving parallel to the axis of the tube the pressure must be the same all over a cross section of the tube; hence Π does not depend upon r. Again, v must be the same for all points at the same distance from the axis, if the fluid is incompressible, for if v changed as we moved parallel to the axis down the tube, the volume of liquid flowing into the ring through P and Q would not be the same as that flowing out. Since Π does not depend upon r, and the left-hand side of equation (1) does not depend upon anything but r, we see that Π must be constant; hence, integrating (1), we get

$$\eta r\frac{dv}{dr} = \tfrac{1}{2}r^2\Pi + C$$

where C is a constant; we have therefore

$$\eta\frac{dv}{dr} = \tfrac{1}{2}r\Pi + \frac{C}{r}$$

Integrating again we have

$$\eta v = \tfrac{1}{4}r^2\Pi + C\log r + C' \tag{2}$$

where C′ is another constant of integration. Since the velocity is not infinite along the axis of the tube—*i.e.*, when $r=0$, C must vanish. To determine C′ we have the condition that at the surface of the tube the liquid is at rest, or that there is no slipping of the liquid past the walls of the tube. This has been doubted; indeed, Helmholtz and Piotrowski thought that they detected finite effects due to the slipping of the liquid over the solid. Some very careful experiments made by Whetham seem to show that under any ordinary conditions of flow no appreciable slipping exists, at least in the case of liquids. We shall assume then that $v=0$ at the surface of the tube—*i.e.*, when $r=a$; this condition reduces equation (2) to

$$\eta v = \tfrac{1}{4}(r^2 - a^2)\Pi \tag{3}$$

Now if p_1 is the pressure where the liquid enters the tube, p_2 the pressure where it leaves it, l the length of the tube,

$$\Pi = -\frac{(p_1 - p_2)}{l}$$

the negative sign is taken because the pressure gradient was taken positive when the pressure increases in the direction of v. Substituting this value for Π, equation (3) becomes

$$\eta v = \frac{p_1 - p_2}{4l}(a^2 - r^2) \tag{4}$$

The volume of liquid Q which passes in unit time across a section of

the tube $$= \int_0^a 2\pi r v dr.$$

$$\therefore Q = \frac{(p_1 - p_2)\pi}{8l\eta} a^4 \qquad (5)$$

This is the law discovered by Poiseuille for the flow of liquids through capillary tubes. We see that the quantity flowing through such a tube is proportional to the square of the area of cross-section of the tube.

When the liquid flows through the capillary tube from a large vessel, as in Fig. 159, the pressure p_1 at the orifice A of the capillary tube differs slightly from that due to the head of the liquid above A, for this head of liquid has not merely to drive the liquid through the capillary tube against the resistance due to viscosity, it has also to communicate velocity and therefore kinetic energy to the liquid, so that part of the head is used to set the liquid in motion. We can calculate the correction due to this cause as follows: let h be the height of the surface of the liquid in the large vessel above the outlet of the capillary tube, ρ the density of the liquid; then if Q is the volume of the liquid flowing through the tube in unit time, the work done in unit time is equal to $g\rho h$Q. This work is spent (1) in driving the liquid through the capillary tube against viscosity, and this part is equal to $(p_1 - p_2)$ Q if p_1 and p_2 are the pressures at the beginning and end of the capillary tube (2) in giving kinetic energy to the liquid. The kinetic energy given to the liquid in unit time is equal to

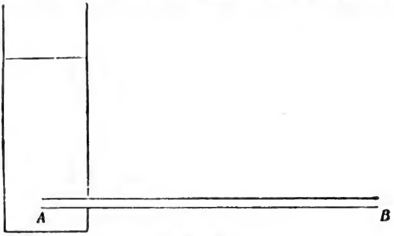

FIG. 159.

$$\tfrac{1}{2}\rho \int_0^a v^2 \times v \times 2\pi r dr$$

where v is the velocity of exit at a distance r from the axis of the capillary tube. If we assume that the distribution of velocity given by equation (4) holds right up to the end B of the tube, then by the help of the equation (5)

we have $$v = \frac{2Q}{\pi a^4}(a^2 - r^2)$$

Substituting this value in the integral we find that the kinetic energy possessed by the fluid issuing from the tube in unit time is $\rho Q^3/\pi^2 a^4$; hence, equating the work spent in unit time to the kinetic energy gained plus the work done in overcoming the viscous resistance, we have

$$g\rho h Q = \frac{\rho Q^3}{\pi^2 a^4} + (p_1 - p_2)Q,$$

or

$$g\rho\left(h - \frac{Q^2}{\pi^2 a^4 g}\right) = p_1 - p_2$$

Thus the head which is spent in overcoming the viscous resistance is not h, but

$$h - \frac{Q^2}{\pi^2 a^4 g}$$

This correction has been investigated by Hagenbach,* Couette,† and Wilberforce,‡ and has been shown to make the results of experiments agree more closely with theory. It is probably, however, not quite accurate on account of the assumption made as to the distribution of velocity at the orifice.

Viscosity of Gases.—The viscosity of gases may be measured in the same way as that of liquids, but the case of a gas flowing through a capillary tube differs somewhat from that investigated on p. 208, where the liquid was supposed incompressible and the density constant; in the case of the gas the density will, in consequence of the variation in pressure, vary from point to point along the tube. Using the notation of the previous investigation, instead of v being constant as we move parallel to the axis of the tube, the fact that equal masses pass each cross-section requires ρv to be constant as long as we keep at a fixed distance from the axis of the tube. Since ρ is proportional to p, where p is the pressure of the gas, we may express this condition by saying that pv must be independent of z where z is a length measured along the axis of the tube. Thus, since p varies along the tube, v will not be constant as z changes; this variation of v will introduce relative motion between parts of the gas at the same distance from the axis of the tube, and will give rise to viscous forces which did not exist in the case of the incompressible liquid. We shall, however, neglect these for the following reasons: if V_0 is the greatest velocity of the fluid, the gradient of velocity along the tube is of the order V_0/l, where l is the length of the tube; the gradient of velocities across the tube is of the order V_0/a, where a is the radius of the tube; as a is very small compared with l, the second gradient, and therefore the viscous forces due to it are very large compared with those due to the first, We shall therefore neglect the effect of the first gradient. On this supposition equation (1) still holds, and, since $\Pi = \frac{dp}{dz}$, we have

$$\eta\frac{d}{dr}\left(r\frac{dv}{dr}\right) = r\Pi = r\frac{dp}{dz}$$

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* Hagenbach, *Poggendorff's Annalen*, 109, p. 385.

† Couette, *Annales de Chimie et de Physique*, [6], 21, p. 433

‡ Wilberforce, *Philosophical Magazine*, (5) 31, p. 407.

or, regarding p as constant over a cross-section of the tube, we have

$$\eta\frac{d}{dr}\left(r\frac{d(pv)}{dr}\right)=rp\frac{dp}{dz}=\tfrac{1}{2}r\frac{dp^2}{dz}$$

Since pv is independent of z, we see that $\frac{dp^2}{dz}$ is constant and equal to $-(p_1^2-p_2^2)/l$.

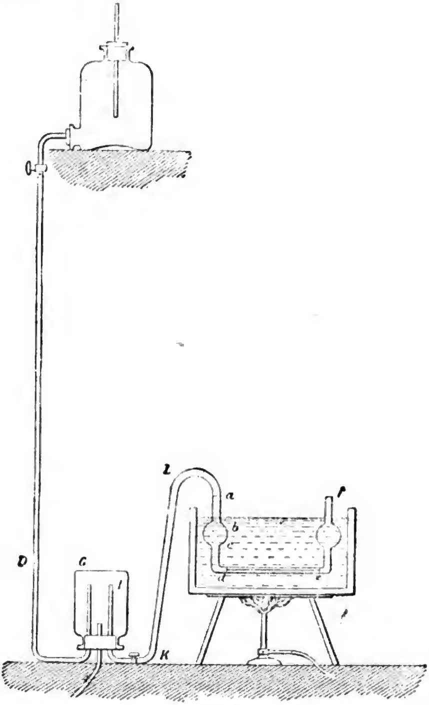

FIG. 160.

Solving the differential equation in the same way as that on p. 208, we get

$$\eta pv=\frac{p_1^2-p_2^2}{8l}(a^2-r^2)$$

and if V_1 is the volume entering, V_2 that leaving the tube per second, we have

$$p_1V_1 = p_2V_2 = \frac{(p_1^2 - p_2^2)}{16l\eta}\pi a^4$$

Measurement of the Coefficient of Viscosity.—The viscosity η has most frequently been determined by measurements of the rate of flow of the fluid through capillary tubes. An apparatus by which this can be done is shown in Fig. 160. *G* is a closed vessel containing air under pressure; the pressure in this vessel is kept constant by means of the tube *D*, which connects *G* with a Mariotte's bottle; the pressure in *G* is always that due to a column of water whose height is the height of the bottom of the air tubes in the Mariotte's bottle above the end of the tube *D*. The glass vessel *abcdef*, in which *de* is a capillary tube, contains the fluid whose coefficient of viscosity is to be determined; this vessel communicates with *G* by means of the tube *LKI;* the pressure acts on the liquid in *abcdef*, and causes it to flow through the capillary tube from left to right; two marks are made at *b* and *c*, and the volume between these marks is carefully determined. Let us call it V; then, if T is the time the level of the liquid takes to fall from *b* to *c*, $Q = V/T$. The area of cross-section of the tube has to be determined with great care, and precautions must be taken to prevent any dust getting into the capillary tube. As the viscosity varies very rapidly with the temperature, it is necessary to maintain the temperature constant; for this purpose the vessel *abcdef* is placed in a bath filled with water.

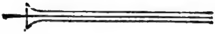

FIG. 161.

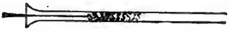

FIG. 162.

With an apparatus of this kind Poiseuille's law can be verified, and the viscosity determined. It is found that, although Poiseuille's law holds with great exactness when the rate of flow is slow, yet it breaks down when the mean velocity $Q/\pi a^2$ exceeds a certain value depending on the size of the tube and the viscosity of the liquid. This point has been investigated by Osborne Reynolds, who finds that the state of flow we have postulated in deducing Poiseuille's law—*i.e.*, that the liquid moves in straight lines parallel to the axis of the tube—cannot exist when the mean velocity exceeds a critical value; the steady flow is then replaced by an irregular turbulent motion, the particles of liquid moving from side to side of the tube. This is beautifully shown by one of Reynolds' experiments. Water is made to flow through a tube such as that shown in Fig. 161, and a little colouring matter is introduced at a point at the mouth of the tube: if the velocity is small the coloured water forms a straight band parallel to the axis of the tube, as in Fig. 161; when the velocity is increased this band becomes sinuous and finally loses all definiteness of outline, the colour filling the whole of the tube, as in Fig. 162. Reynolds concluded from his experiments that the steady motion cannot exist if the mean velocity is greater than 1000 $\eta/\rho a$ where η is the viscosity, ρ the density of the liquid, and a the radius of the tube. The units are centimetre, gramme and second.

Measurements of the viscosity of fluids both liquid and gaseous, have been made by determining the couple which must be applied to a cylinder to keep it fixed when a coaxial cylinder is rotated with uniform velocity, the space between the cylinders being filled with the liquid whose viscosity has to be determined. This method has been used by Couette and Mallock. The theory of the method is as follows: the particles of the fluid will describe circles round the common axis of the cylinders. Let PQ be points on a radius of the cylinders; after a time T, let P come to P', Q to Q', let OP' produced cut QQ' in Q''. Then the velocity gradient at P will be equal to $(Q'Q''/\mathrm{T}) \div P'Q''$; if ω is the angular velocity with which the particle at P describes its orbit, $\omega + \delta\omega$ that of the particle at Q, then $Q'Q'' = OQ'\delta\omega\mathrm{T}$. Let $OP = r$, $OQ = r + \delta r$, then since $P'Q'' = \delta r$ the velocity gradient at P is $(r + \delta r)\dfrac{\delta\omega}{\delta r}$, or when δr is very small, $r\dfrac{d\omega}{dr}$; hence the tangential stress acting on unit area of the surface at P is $\eta r\dfrac{d\omega}{dr}$. Now consider the portion of liquid bounded by coaxial cylinders through P and R and by two parallel planes at right angles to the axes of the cylinders and at unit distance apart. This annulus is rotating with constant angular velocity round the axis of the cylinders, hence the moment about this axis of the forces acting upon the annulus must vanish. Now the moment of the forces acting on the inner face of this annulus is

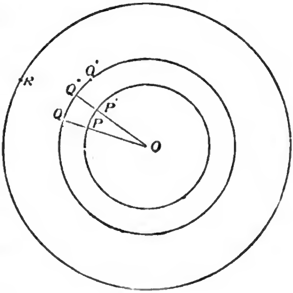

FIG. 163.

$$2\pi r\eta r\frac{d\omega}{dr}r = 2\pi\eta r^3\frac{d\omega}{dr}$$

and this must be equal and opposite to the moment of those acting on the outer surface of the cylinder; now R may be taken anywhere; hence we see that this expression must be constant and equal to the moment of the couple acting on unit length of the outer cylinder, which is, of course, equal and opposite to the moment of that on the inner. Let us call this moment

Γ, then

$$2\pi\eta r^3\frac{d\omega}{dr} = \Gamma$$

Integrating this equation we find

$$\omega = -\frac{\Gamma}{4\pi\eta r^2} + C$$

where C is a constant. If the radii of the inner and outer cylinders are a and b respectively, and if the inner cylinder is at rest and the outer one rotates with an angular velocity Ω, then since $\omega = 0$, when $r = a$, and $\omega = \Omega$

when $r = b$, we find

$$\Gamma = 4\pi\eta\frac{a^2b^2}{b^2 - a^2}\Omega$$

Hence, if we measure Γ for a given velocity Ω, we can deduce the value of η. This case presents the same peculiarities as the flow of a viscous liquid through a capillary tube; the law expressed by the preceding equation is only obeyed when Ω is less than a certain critical value. When Ω exceeds this value the motion of the fluid becomes turbulent, and for values of Ω just above this value the relation between Γ and Ω becomes irregular; it becomes regular again when Ω becomes considerably greater, but Γ is no longer proportional to Ω, but is of the form $\alpha\Omega+\beta\Omega^2$ where α and β are constants. These facts are well shown by the curve given in *Fig.* 164, which represents the results of Couette's* experiments on the viscosity of water. The abscissæ are the values of Ω and the ordinates the values of Γ/Ω. The instability set in at **B** when the outer cylinder made about one revolution per second; the radii of the cylinders were 14·64 and 14·39 cm. respectively.

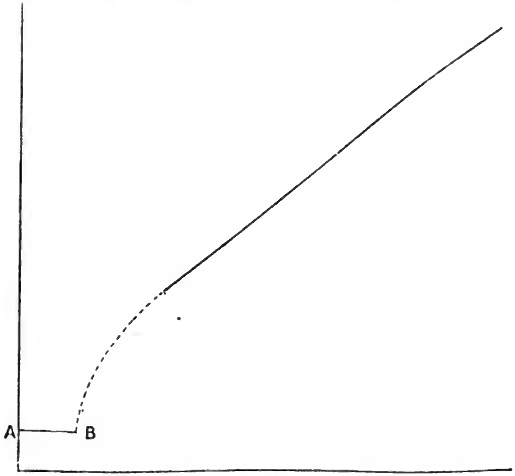

FIG. 164.

This method can be applied to determine the viscosity of gases as well as of liquids.

Method of the Oscillating Disc.—Another method of determining η, which has been used by Coulomb, Maxwell, and O. E. Meyer, is that of measuring the logarithmic decrement of a horizontal disc vibrating over a fixed parallel disc placed at a short distance away, the space between the discs being filled with the liquid whose viscosity is required. The viscosity

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[*] Couette, *Annales de Chimie et de Physique* [6], 21, p. 433.

of the liquid gives rise to a couple tending to retard the motion of the disc proportional to the product of the angular velocity of the disc and the viscosity of the liquid: the calculation of this couple is somewhat difficult. We shall refer the reader to the solution given by Maxwell

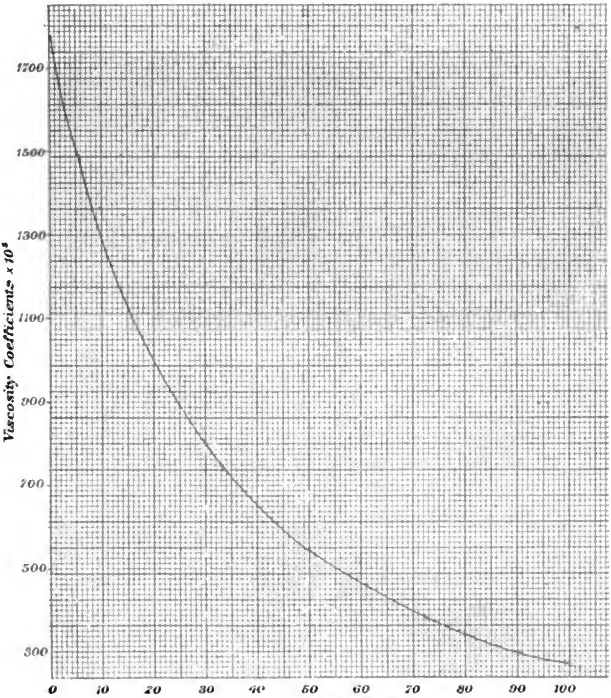

FIG. 165.

(*Collected Papers*, vol. ii. p. 1). This method, as well as the preceding one, can be used for gases as well as for liquids.

Among other methods for measuring η we may mention the determination of the logarithmic decrement for a pendulum vibrating in the fluid (Stokes); the logarithmic decrement of a sphere vibrating about a diameter in an ocean of the fluid; the logarithmic decrement of a hollow sphere filled with the liquid and vibrating about a diameter (Helmholtz and Piotrowski, *Helmholtz Collected Papers*, vol. i. p. 172).

Temperature Coefficient of Viscosity.—In all experiments on viscosity it is necessary to pay great attention to the measurement of the temperature, as the coefficient of viscosity of liquids diminishes very rapidly as the temperature increases. This is shown by the curve (Fig. 165) taken from the paper by Thorpe and Rodger (*Phil. Trans.*, 1894, A. Part ii. p. 397), which shows the relation between the viscosity of water and its temperature. It will be seen that the viscosity of water at 80° C. is only about one-third of its value at 10° C. Thorpe and Rodger, who determined the co-efficients of viscosity of a large number of liquids, found the formula given by Slotte, $\eta = C/(1+bt)^n$, where η is the co-efficient of viscosity at the temperature t and C, b and n are constants depending on the nature of the liquid, was the one that agreed best with their experiments. For water they found that

$$\eta = \frac{\cdot 017941}{(1+\cdot 023120t)^{1\cdot 5423}}$$

where t is the temperature in degrees Centigrade.

The following table, taken from Thorpe and Rodger's paper (*Phil. Trans.*, A. 1894, p. 1), gives the value of η in C.G.S. units for some liquids of frequent occurrence. The table gives the value of the constants C, b, n in Slotte's formula

$$\eta = C/(1+bt)^n$$

SUBSTANCE	C	b	n
Bromine	·012535	·008935	1·4077
Chloroform	·007006	·006316	1·8196
Carbon tetrachloride	·013466	·010521	1·7121
Carbon bisulphide	·004294	·005021	1·6328
Formic acid	·029280	·016723	1·7164
Acetic acid	·016867	·008912	2·0491
Ethyl ether	·002864	·007332	1·4644
Benzene	·009055	·011963	1·5554
Toluene	·007684	·008850	1·6522
Methyl alcohol	·008083	·006100	2·6793
Ethyl alcohol	·017753	·004770	4·3731
Propyl alcohol	·038610	·007366	3·9188
Butyl alcohol:			
0° to 52°	·051986	·007194	4·2452
52° to 114°	·056959	·010869	3·2150
Inactive amyl alcohol:			
0° to 40°	·085358	·008488	4·3249
40° to 80°	·093782	·012520	3·3395
80° to 128°	·152470	·026540	2·4618
Active amyl alcohol:			
0° to 35°	·111716	·009851	4·3736
35° to 73°	·124788	·015463	3·2542
73° to 124°	·147676	·127583	2·0050
Allyl alcohol	·021736	·009139	2·7925
Nitrogen peroxide	·005267	·007098	1·7349

Warburg found that η for mercury at 17·2° is equal to ·016329. A later determination by Umani (*Nuov. Cim.* [4] 3, p. 151) gives $\eta = \cdot 01577$ at 10°.

The value of η for liquid carbonic acid is very small, being at 15° only 1/14·6 of that of water.

Effect of Pressure on the Viscosity.—The viscosity of water diminishes slightly under increased pressure, while that of benzol and ether increases.

Viscosity of Salt Solutions.—A large number of experiments have been made on the viscosity of solutions, but no simple laws connecting the viscosity with the strength of the solution have been arrived at. In some cases the viscosity of the solution is less than that of water, and in many cases the viscosity of the solution is a maximum for a particular strength.

Viscosity of Mixtures.—Here again no general results have been arrived at, although considerable attention has been paid to this subject. In many cases the viscosity of a mixture of two liquids A, B is less than that calculated by the formula

$$\eta = \frac{a\eta_A + b\eta_B}{a+b}$$

where η_A, η_B are respectively the viscosities of A and B, and a, b are the volumes of A and B in a volume $a+b$ of the mixture.

Lubrication.—When the surfaces of two solids are covered with oil or some other lubricant they are not in contact, and the friction between them, which is much less than when they are in contact, is due to fluid friction. The laws of fluid friction discussed in this chapter show that, if we have two parallel planes at a distance d apart, the interval between them being filled with a liquid, then if the lower plane is at rest and the upper one moving parallel to the lower one with the velocity V, if V is not too great there is a retarding tangential force acting on the moving plane, and equal per unit area to $\eta V/d$, where η is a quantity called the coefficient of viscosity of the liquid. If we regard this as a frictional force acting on the moving plate we see that the friction would depend upon the velocity, and would only depend upon the pressure between the bodies in so far as the pressure affected the thickness of the liquid layer and the viscosity of the lubricant.

The laws of friction, when lubricants are used, are complicated, depending largely upon the amount of lubrication. When the lubricant is present in sufficiently large amounts to fill the spaces between the moving parts the friction seems to be proportional to the relative velocity of these parts. When the supply of lubricant is insufficient, part of it collects as a pad between the moving parts, as in Fig. 166; here the lower surface is at rest and the upper one rotating from left to right. Professor Osborne Reynolds[*] has shown that, as the breadth and thickness of this pad depend upon the pressure and relative velocity, it would be possible to get friction proportional to the pressure and independent of the relative velocity, even when the friction was entirely caused by the viscosity of a thin layer of liquid between the moving parts.

Viscosity of Gases.—Gases possess viscosity, and the forces called into play by this property are, as in the case of liquids, proportional to the velocity gradient; in fact, the definition of viscosity given on p. 205, applies to gases as well as to liquids. The most remarkable property of the viscosity of gases is that within wide limits of pressure the viscosity is independent of the pressure, being under ordinary circumstances the same at a pressure of a few millimetres of mercury as at atmospheric pressure. This is known as Maxwell's Law, as it was deduced by Maxwell from the Kinetic Theory of Gases; it has been verified by numerous experiments. Boyle has some claim to be regarded as the discoverer of this law, for about 1660 he experimented on the effect of diminishing the pressure on the vibrations of a pendulum, and found that the vibrations died away just as quickly when the pressure was low as when it was high. This law follows very readily from the view of viscosity supplied by the Theory of Gases. Thus, suppose we have two layers of gas *A* and *B* at the same pressure, and that *A* has a motion as a whole from left to right, while *B* is either at rest or moving more slowly than A in this direction. According to the Kinetic Theory of Gases, molecules of the gas will be continually crossing the plane separating the layer *A* from the layer *B*. Some of these molecules will cross the plane from *A* to *B* and an equal number, since the pressure of the gas remains uniform, from *B* to *A*. The momentum parallel to the plane of those which leave *A* and cross over to *B* is greater than that of those which replace them coming over from *B* to *A*; thus the layer *A* is continually losing momentum while the layer *B* is gaining it. The effect is the same as if a force parallel to the plane of separation acted on the layer *A*, so as to tend to stop the motion from left to right, while an equal and opposite force acted on *B*, tending to increase its motion in this direction; these forces are the viscous forces we have been discussing in this chapter. If the distribution of velocity remains the same, the magnitude of these forces will be proportional to the number of molecules which cross the plane of separation in unit time.

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[*] Reynolds, *Phil. Trans.*, 1886, pt. i. p. 157.

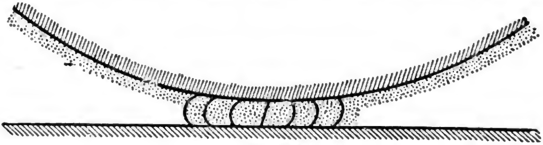

FIG. 166.

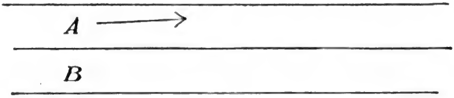

FIG. 167.

The molecules are continually striking against each other, the average free run between two collisions, called the **mean free path of the molecules,** being extremely small, only about 10^{-5} cm. for air, at atmospheric pressure. This free path varies, however, inversely as the pressure, and at the extremely low pressures which can be obtained with modern air-pumps can attain a length of several centimetres. When one molecule strikes against another its course is deflected, so that, although it is travelling at a great speed, it makes but little progress in any assigned direction. The consequence of this is that the molecules which cross in unit time the plane of separation between *A* and *B* can all be regarded as coming from a thin layer of gas next this plane, a definite fraction of the molecules in this layer crossing the plane. The longer the free path of the molecules the thicker the layer, the thickness being directly proportional to the mean free path. If *n* is the number of molecules per unit volume and *t* the thickness of the layer, the number of molecules which in unit time cross unit area of the plane separating *A* and *B* will be proportional to *nt*. Let us consider the effect on this number of halving the pressure of the gas. This halves *n* but doubles *t*; *t* is proportional to the free path, which varies inversely as the pressure, hence the product *nt*, and therefore the viscosity, remains unaltered. This reasoning holds until the thickness of the layer from which the molecules cross the plane of separation gets so large that the layer reaches to the sides of the vessel containing the gas. When this is the case no further diminution in the pressure can increase *t*, and as *n* diminishes as the pressure diminishes, the product *nt* and, therefore, the viscosity, will fall as the pressure falls. Thus in a vessel of given size the viscosity remains unaffected by the pressure until the pressure reaches a certain value, which depends upon the size of the vessel and the nature of the gas; when this pressure is passed the viscosity diminishes rapidly with the pressure. This is shown very clearly by the curves in Fig. 168, based on experiments made by Sir William Crookes (*Phil. Trans.*, 172, pt. ii. 387). In these curves the ordinates represent the viscosity and the abscissæ the pressure of the gas.

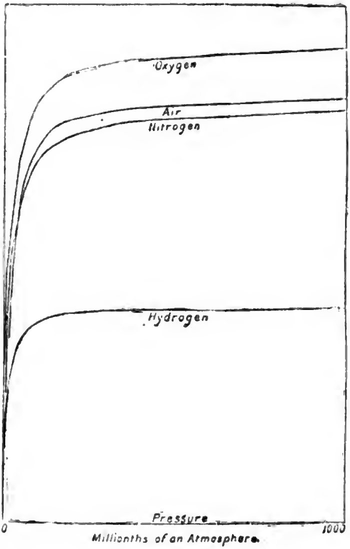

FIG. 168.

The diminution in viscosity at low pressures is well shown by an incandescent electric lamp with a broken filament. If this be shaken while the lamp is exhausted it will be a long time before the oscillations die away; if, however, air is admitted into the lamp through a crack made with a file the oscillations when started die away almost immediately.

Another reason why the effects of viscosity are less at very low pressures than at higher ones is the slipping of the gas over the surface of the solids with which it is in contact. In the case of liquids, no effects due to slip have been detected. Kundt and Warburg[*] have, however, detected such effects in gases even up to a pressure of several millimetres of mercury. The law of slip (*see* Maxwell, "Stresses in a Rarefied Gas," *Phil. Trans.*, 187) may be expressed by saying that the motion in the gas is the same as if a certain thickness L were cut off the solids, and that the gas in contact with this new surface were at rest. This thickness L is proportional to the mean free path of the molecules of the gas. According to the experiments of Kundt and Warburg it is equal to twice the free path; hence, as soon as the free path gets comparable with the distance between the solids in the gas, the slip of the gas over these solids will produce appreciable effects in the same direction as a reduction in viscosity.

Mean Free Path.—If we know the value of the viscosity we can calculate the mean free path of the molecules of a gas: for if we calculate, from the principles of the Kinetic Theory of Gases, the rate at which momentum is flowing across unit area of the plane *A*, *B*, Fig. 167, we find that it is equal to

$$\cdot 350 c\rho\lambda \frac{dv}{dx}$$

where v is the velocity of the stratum at a height x above a fixed plane, λ is the mean free path, ρ the density of the gas, c the "velocity of mean square" (this can be calculated from the relation $p = \frac{1}{3}\rho c^2$ where p is the pressure in the gas). The rate of flow of momentum across unit area is equal to the tangential stress at the plane AB; hence, if η is the viscosity of the gas, $\eta = \cdot 350 c\rho\lambda$. Let us calculate from this equation the value of λ for air; taking for the viscosity at atmospheric pressure and at 15° C. $\eta = 1{\cdot}9 \times 10^{-4}$, ρ at pressure 10^6 and temperature 15° C., $1{\cdot}26 \times 10^{-3}$, we get $c = 4{\cdot}88 \times 10^4$, and $\lambda = \cdot 00001$ cm. At the pressure of a millionth of an atmosphere the mean free path in air is 10 cm.

The values of η for a few of the most important gases are given in the following table; the temperature is about 15° C. These numbers are given by O. E. Meyer; they are deduced from his own experiments on the viscosity of air by the method of the oscillating disc and the experiments made by Graham on the relation between the rates of flow of different gases through capillary tubes:

Gas	$\eta \times 10^4$	Gas	$\eta \times 10^4$
Air	1·9	Sulphuretted hydrogen	1·3
Hydrogen	·93	Hydrochloric acid	1·56
Marsh-gas	1·2	Carbonic acid	1·6
Water-vapour	·975	Nitrous oxide (N_2O)	1·6
Ammonia	1·08	Methyl ether	1·02
Carbonic oxide	1·84	Methyl chloride	1·16
Ethylene	1·09	Cyanogen	1·07
Nitrogen	1·84	Sulphurous acid (SO_2)	1·38
Oxygen	2·12	Ethyl chloride	1·05
Nitric oxide (NO)	1·86	Chlorine	1·41

---

[*] *Pogg. Ann.*, 155, p. 357.

Effect of Temperature upon the Viscosity of Gases.—Increase of temperature has opposite effects on the viscosities of liquids and of gases, for while, as we have seen, it diminishes the viscosity of liquids it increases that of gases. If η is the coefficient of viscosity, and if this is assumed to be proportional to T^n where T is the absolute temperature, then, according to Lord Rayleigh's[*] experiments, we have the following values for n:

	n	c
Air	·754	111·3
Oxygen	·782	128·2
Hydrogen	·681	72·2
Helium	·681	72·2
Argon	·815	150·2

The values of c relate to a formula suggested by Sutherland, according to which $\eta = a\frac{T^{\frac{1}{2}}}{1+c/T}$; thus, at very high temperatures, if this relation is true, η would vary as the square root of the absolute temperature. According to Koch,[†] the viscosity of mercury vapour varies much more rapidly with the temperature than that of any other known gas. He concluded from his experiments that for this gas $\eta = aT^{1·6}$. The results given above for helium and argon, both, like mercury vapour, monatomic elements, show that a rapid variation with temperature is not a necessary characteristic of monatomic gases. Lord Rayleigh found that the viscosity of argon was 1·21, and of helium 0·96 that of air.

Coefficient of Viscosity of Mixtures.—Graham made an extensive series of experiments on the coefficients of viscosity of mixtures of gases by measuring the time taken by a known volume of gas to flow through a capillary tube. He found that for mixtures of oxygen and nitrogen, and of oxygen and carbonic acid, the rate of flow through the tubes of the mixture was the arithmetical mean rate of the gases mixed; with mixtures containing hydrogen the results were very different; how different is shown by the following table, which gives the ratio of the transpiration time of the mixtures to that of pure oxygen:

Hydrogen and Carbonic Acid.			Hydrogen and Air.		
100	0	·4321	100	0	·4434
97·5	2·5	·4714	95	5	·5282
95	5	·5157	90	10	·5880
90	10	·5722	75	25	·7488
75	25	·6786	50	50	·8179
50	50	·7339	25	75	·8790
25	75	·7535	10	90	·8880
10	90	·7521	5	95	·8960
0	100	·7470	0	100	·900

It will be seen from this table that, while the addition of 5 per cent. of air to pure hydrogen alters the time of effusion by about 20 per cent., the mixture of half hydrogen, half air, has a time of effusion which only differs from that of pure air by about 8 per cent. Thus the addition of hydrogen to air has little influence on the viscosity, while the addition of air to hydrogen has an enormous influence.

Resistance to a Solid moving through a Viscous Fluid.—When a solid moves through a fluid the portions of the fluid next the solid are moving with the same velocity as the solid, while the portions of the fluid at some distance off are at rest. The movement of the solid thus involves relative motion of the fluid; the viscosity of the fluid resists this motion, so that there is a force acting on the solid tending to resist its motion.

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[*] Rayleigh, *Proc. Roy. Soc.*, 66, p. 68.

[†] Koch, *Wied. Ann.*, 19, p. 587.

Sir George Stokes has shown that in the case of a sphere moving with a very small uniform velocity V through the fluid the force resisting the motion is equal to $6\pi\eta a V$ where a is the radius of the sphere, η the viscosity of the fluid through which it is falling. Consider now the case of a sphere falling through a viscous fluid; just after starting from rest the velocity will be small and the weight of the sphere will be greater than the viscous resistance; the velocity of the sphere, and therefore the resistance, will increase until the resistance is equal to the weight of the sphere. When this velocity, which is called the **critical velocity**, is reached, the forces acting on the sphere will be in equilibrium, and the sphere will fall with a uniform velocity which may also be called the **terminal velocity**. Since the effective weight of the sphere is equal to $4\pi a^3(\rho - \sigma)g/3$, where ρ is the density of the sphere and σ that of the liquid through which it is moving, if V is the terminal velocity,

$$6\pi\eta a V = \frac{4\pi}{3}a^3(\rho - \sigma)g$$

or

$$V = \frac{2}{9}\frac{ga^2(\rho - \sigma)}{\eta} \qquad (1)$$

so that the terminal velocity is proportional to the square of the radius of the sphere. In the case of a drop of water falling through air for which $\eta = 1{\cdot}8 \times 10^{-4}$, we find, if the radius of the drop is 1/100 of a millimetre, $V = 1{\cdot}2$ cm./sec. This result explains the slow rate at which clouds consisting of fine drops of water fall. Since η is independent of the pressure, the terminal velocity in a gas will, since σ in this case is small compared with ρ, be independent of the pressure.

As an application of this formula we may mention that the size of small drops of water has been determined by measuring the rate at which they fell through air; from this the value of the radius can be determined by equation (1). The expression for the resistance experienced by the sphere falling through the viscous liquid is obtained on the supposition that the motion of the liquid is so slow that terms depending upon the squares of the velocity of the liquid can be neglected in comparison with those retained. Now, if V is the velocity, ρ the density of the liquid, the forces on the liquid depending upon the squares of the velocity, are proportional to the gradient of the kinetic energy per unit volume—*i.e.*, to the gradient of $\frac{1}{2}\rho V^2$; the forces due to viscosity are proportional to the gradient of the viscous stress. If a is the radius of the sphere, the distance from the sphere at which the velocity may be neglected is proportional to a, hence the velocity gradient is of the order (V/a), and the viscous stress $\eta V/a$. Hence, if we can reject the effects depending on the squares of the velocity in comparison with the effects of viscosity, ρV^2 must be small compared with $\eta V/a$, or $\rho V a$ must be small compared with η. Hence, if the preceding solution holds, we see, by substituting for V the value of the limiting velocity, that $\frac{2}{9}g\frac{a^3(\rho - \sigma)\rho}{\eta^2}$ must be small. Lord Rayleigh[*]

---

[*] Lord Rayleigh, *Phil. Mag.*, [5] 36, p. 354.

has pointed out how much this restricts the application of Stokes' result; thus, for example, in the case of drops of water falling through air, the theory does not apply if the drops are more than about one-tenth of a millimetre in radius. When the velocity of the falling body exceeds a certain critical value the motion of the surrounding fluid becomes turbulent, just as when the velocity of a fluid through a capillary tube exceeds a certain value the flow ceases to be regular (*see* p. 212). When this turbulent stage is reached the resistance becomes proportional to the square of the velocity. Mr. Allen,[*] who has recently investigated the resistance experienced by bodies falling through fluids, finds that this can be divided roughly into three cases—(*a*) where the velocity is very small, when the preceding theory holds, and the resistance is proportional to the velocity; (*b*) a stage where the velocity is great enough to make the forces depending on the square of the velocity comparable with those depending on viscosity; in this stage the resistance is proportional to the velocity raised to the power of $3/2$; (*c*) a stage where the velocity is so great that the motion of the fluid becomes turbulent; in this stage he finds the resistance to be proportional to the square of the velocity. When the resistance is proportional to the square of the velocity the method of dimensions shows that it does not for a given velocity depend upon the viscosity of the liquid. For, suppose the resistance is proportional to $a^x \rho^y \eta^z V^n$, this expression must be of the dimensions of a force—*i.e.*, 1 in mass, 1 in length, and -2 in time; hence we have

$$\begin{aligned} 1 &= y + z \\ 1 &= x - 3y - z + n \\ -2 &= -z - n \end{aligned}$$

so that
$$x = n,\ y = n - 1,\ z = 2 - n,$$

and the resistance is proportional to $(Va\rho/\eta)^n(\eta^2/\rho)$; thus, if $n = 2$ the resistance is proportional to $V^2a^2\rho$, and is independent of viscosity. The energy of the body is spent in producing turbulent motion in the liquid and not in overcoming the viscous resistance.

A great deal of attention has been given to the resistance of bodies moving with high speeds, such as bullets. It is doubtful, however, if the viscosity of the fluid through which the bullet moves has any effect upon the resistance; we shall not, therefore, enter into this subject, except to say that the most recent researches, those by Zahm, seem to indicate that for velocities less than about 30000 cm./sec. the resistance may be represented by $av^2 + bv^3$, where a and b are constants.

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[*] Allen, *Phil. Mag.*, Sept. and Nov. 1900.

INDEX

ACCELERATION due to gravity, 7–24
Air, deviations from Boyle's law as to, 126
Airy, hydrostatic theory of earth's crust, 23
  Dolcoath experiment, 35
  Harton pit experiment, 35
Amagat, minimum value of pv., 126, 127
Angle of shear, 66
Arc, correction for pendulum swing, 10
Atmolysis, 202

BAILY'S Cavendish experiment, 39
Baille and Cornu's experiment, 39
Bars, bending of, 85–102
  vibration of, 94
Barymeter, von Sterneck's, 26
Bending of rods or bars, 85–102
Bernouilli's correction for arc of swing of pendulum, 10
Boiling-point, depression of, in solutions, 191
Borda's pendulum experiments, 10
Bouguer's pendulum experiments, 10
  experiments on determination of density of earth, 32
  rule and exceptions, 22–3
Boyle's law, 125
  at low pressures, 128
  deviations of various gases from, 126
Boys's Cavendish experiment, 40
Braun's Cavendish experiment, 41
Breaking-point of stretched wires, 55
Bubbles and drops, measurement of surface tension by, 156, 161

CAMPHOR, movements of on surface of water, 169
Capillarity, 135–181
  Laplace's theory of, 173–181
Capillary tubes, rise of fluids in, 140
Carbonic acid, deviation of, from Boyle's law, 126
Carlini's pendulum experiment, 35
Cassini's and Borda's pendulum experiment, 10
Cavendish experiment, 36
  by other observers, 39
  see Earth, determination of density of
Clairaut's theorem, 22

Collision, 109
  duration of, on impact, 112
  of drops, 172
  see also *Impact*
Colloids, 186
Compressibility of liquids, *see Liquids*
Computed times of pendulums, 15
Contamination of films, 170
Critical velocity in viscous fluids, 222
Crystalloids, 186

DEFFORGES' pendulum, 19
Degree of latitude, measurement of a, 21
Diaphragm, diffusion through, 186, 200
Differential gravity balance, 26
Diffusion of gases, *see Gases*
  of liquids, *see Liquids*
  of metals, 204
Dilatation under strain, 64
Dissociation of electrolytes, 194

EARTH, determination of density of, 31
  by Airy, 35
    Baily, 39
    Bouguer, 32
    Boys, 41
    Braun, 41
    Carlini, 35
    Cavendish, 36
    Cornu and Baille, 39
    von Jolly, 42
    Maskelyne, 33
    Mendenhall, 35
    Poynting, 43
    Richarz and Krigar-Menzel, 42
    von Sterneck, 36
    Wilsing, 41
Effusion, thermal, 202
Elastic after-effect, 55
  curve, 95
  fatigue, 57
  limit, 53, 69
Elasticity, 53
  modulus of, 69, 102
  see also *Young's Modulus*
Electrolytes, dissociation of, 194
Ellipticity of earth, 23, 24
Elongation under strain, 64

Equilibrium of liquids in contact, 139
Equivalent simple pendulum, 13

FATIGUE, elastic, 57
Faye's rule, 23
Films, contamination of, 170
  cooling effects, on stretching, 163
  stability of cylindrical, 147
Flexure, 99
Floating bodies, forces acting on, 153
Fluid motion, effect of, on pendulums, 14
  surfaces, disruption of, 174
Formulæ for pendulum motion, 13–24
Freezing-point, depression of in solutions, 193

GALILEO'S observations respecting pendulums, 8
Gaseous pressures and volumes, 124
Gases, diffusion of, 196
  kinetic theory as applied to the, 198
  obstruction to, offered by perforated diaphragms, 200
  through porous bodies, 201
Gases, passage of, through india-rubber, 203
  through liquids, 203
  through red-hot metals, 204
Gases, viscosity of, 210, 218
  influence of temperature upon, 221
Gravitation, constant, 29
  Newton's law, 28
  qualities of, 45–52
  see also Earth, density of
Gravity, acceleration of, 7
  history of research, as to, 7
  Clairaut's theorem, 22
  Newton's theory of, 20
  Richer's observations on, 20
  Swedish and Peruvian expeditions of investigation, 21
Gravity balance, Threlfall and Pollock's, 27
Gravity meters, differential, 26

HALF-SECONDS pendulum, von Sterneck, 24
Hodgkinson's table of values of *e* on impact, 114
Homogeneous strain, 62
Hooke's law, 69
Hydrogen, deviations of, from Boyle's law, 126
Hydrostatic theory, 23
Huygens' pendulum clock, 9
  theory of pendulums, 9

INDIAN survey, experiments on pendulums, 23
Impact, 109
  duration of collision on, 112
  kinetic energy of, 110
Invariable pendulum, 22

JAEGER'S method of determining mean surface-tension, 162
Jolly, von, experiments on gravitation 42

KATER'S convertible pendulum, 12
  and Sabine's experiments, 23
Kelvin's table of thermal effects accompanying strain, 134
Kinetic theory of gases, 218
  explanation of diffusion by the, 198

LAPLACE'S theory of capillarity, 173
Latitude, determination of length of 1° of, 21
Liquids, capillarity of, 135
  compressibility of, 116, 122
  diffusion of, 183
    determination of co-efficient of 184
    through membranes, 186
  in contact, 139
  films, stability of, 147
  flow of viscous, through cylindrical capillary tubes, 207
  potential energy of, due to surface tension, 137
  rise of, in capillary tubes, 140
  surface-tension of, 137
    relation between curvature and pressure of surface, 142
    methods of measuring, 155
      by bubbles and drops, 156, 161
      by ripples, 157
    temperature, effects on, 163
  table of compressibility of various, 122
  tensile strength of, 122
  vapour-pressure over curved surface of, 166
  viscosity of, 205
Loaded pillar, stability of, 97
  wires, anomalous effects in, 58
Lubrication, 217

MASS, 3
  constancy of, 5
  definition of, 4
  unit of, 5
Maxwell's law of gaseous viscosity, 218
Mean free path, 218, 220
Mendenhall's gravitation experiment, 35
Mercury, compressibility of, 121
Metals, diffusion of, through metals, 204
  elastic properties of, 53, 57
  viscosity of, 57
Michell, Rev. J., 36
Microstructure of metals under stress 58
Modulus of elasticity, 69, 102
    Young's, 70, 73, 74, 76
  of rigidity, 7

Newton's theory of gravitation, 28
  theory of gravity, 20
Nitrogen, deviation of, from Boyle's law, 126
Normal stress, 68

Oil, effect of, on waves, 171
Osmosis, 186
Osmotic pressure, 188

Pendulums, Bessel's experiments, 13
  Borda and Cassini's, 10
  clock, 9
  Defforges, 19
  formulæ for, 13–24
  Half-seconds pendulum, 24
  Huygens' theory of, 9
  Indian survey experiments, 23
  invariable, 23
  Kater's convertible, 12
  Newton's use of, 9
  Papers on the theory of, 7
  Repsold's, 18
  von Sterneck's, 24
  U.S. survey, 20
  variation in length of seconds, 2
  yielding of support of, 18
Permanent set, 53
Picard's pendulum experiments, 9
Piezometer (the), 119
Poiseuille's law, 209
Poisson's ratio, 73, 87, 120
Poynting's gravitation experiments, 43
Pressure, effect of, on viscosity, 217, 219
    on volume, 124
  variations from Boyle's law at low, 128

Quartz thread gravity balance, Threlfall's, 27

Reich's Cavendish experiment, 39
Repsold's pendulum, 18
Resolution of strain, 65
Reversible pendulum, theory of, 13
Reversible thermal effects accompanying strain, 131
Richer, observations on gravity, 20
Rigidity, co-efficient of, 83
  modulus of, 70
Ripples, measurement of surface-tension by, 157
Rods, stresses and strains of, 71, 73, 79, 83, 85–102

Sabine's pendulum, 23
Salt solutions, viscosity of, 217
Schiehallion experiment, 32
Shear, 65
  angle of, 66
Soap-bubbles, 143
Solutions, depression of boiling-point of, 191
    of freezing-point of, 193
  vapour pressure of, 190
Spiral springs, 101–108
  energy of, 104–108
Stability of cylindrical films, 147
  of loaded pillar, 97
Sterneck, von, Barymeter, 26
  half-seconds pendulum, 24
  pendulum experiments, 36
Strain, 62
  anomalous effects of alternating, on wire, 58
  alteration of micro-structure consequent on, 59
  axes of, 64
  homogeneous, 62
    resolution of a, 65
  in relation to work, 70
  thermal effects accompanying, 131
Stresses, 68
  on bars, 71
Stretched film, 144
  cooling due to stretching, 163
Stretched wire, anomalous effects on loading, 58
Surface-tension, 137
  effects between two liquids, 179
    in thick films, 178
  forces between 2 plates, due to, 152
Surface-tension, Jaeger's method of measuring, 162
  oscillations of a spherical drop under, 160
  of thin films, 164
  measurement of by detachment of a plate, 161
  Ripple method, 157
  Wilhelmy's method, 161
Swedish and Peruvian expeditions to determine length of 1° of latitude, 21

Table of moduli of elasticity, 102
  thermal effects of strain, 131
Tangential stress, 68
Temperature, co-efficient of viscosity, 216
  effects of, on surface-tension, 163
    on breaking stress of wires, 61
Tensile strength of liquids, 123
Terminal velocity in viscous fluids, 222
Thermal effects of strain, 131
  Kelvin's table of, 134
Thermal effusion, 202
Thickness of films, influence of, on surface-tension, 178
Thin films, surface-tension of, 114
Threlfall and Pollock's gravity balance, 27
Torsion, 78
  in cylindrical tubes, 78
  in solid rods, 79

U.S. SURVEY pendulums, 20

VAPOUR, diffusion of, 197
Vapour pressure, of solutions, 190
  on curved surfaces, 166
Vibration of bars, 95
Viscosity, 60
  temperature co-efficient of, 216
  determination of co-efficient of, 212
    by oscillating disc, 214
  effects of pressure upon, 217
  gaseous, effect of temperature on, 221
  of gases, 218
  of liquids, 205
  of metals, 57
  of mixtures, 221
  of salt solutions, 217
Viscous fluids, resistance of, to motion of solids, 221
  velocity in, 222
Volume and pressure of gases, 124

WATER, compressibility of, 121
Waves, calming of, by oil, 171
Weight, 1
  standards of, 5
Wilhelmy's method of measuring surface-tension, 162
Wilsing's gravitation experiments, 41
Work in relation to strain, 70

YIELD point, 53
Young's modulus, 70, 73
  determination of, 74
    by flexure, 99
    by optical measurement, 76

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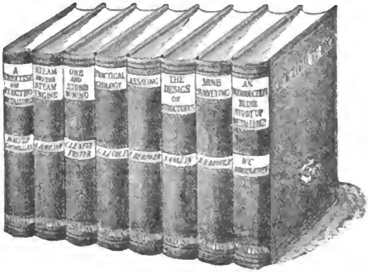

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PAGE

ADAMS (W. P.), Motor Mechanism, . 33
AITKEN, (T.), Road Making, . . 79
ALFORD (J.), Mining Law, . . 57
ALLINGHAM (W.), Meteorology, . 41
ANDREWS (L.), Electricity Control, 48
ANGLIN (S.), Design of Structures, . 26
ARCHBUTT & DEELEY, Lubrication, 32
ATHERTON (Wm.), Design of Beams, 27
BARKER (D. WILSON-), Navigation, 41
—— Seamanship, . . 40
BAUER (M.), Precious Stones, . 68
BERINGER (J. J. & C.), Assaying, . 66
BICHEL & LARSEN, Explosives, . 58
BILES (Prof.), Marine Steam Turbine, 29
BJÖRLING, Peat, ... 75
BLACKMORE (E.), Mercantile Marine, 40
BLOUNT (B.) and BLOXAM (A. G.), Chemistry for Engineers, . . 46
—— Chemistry for Manufacturers, 71
BLYTH (A. Wynter), Foods and Poisons, 72
BORCHERS (Dr.), Electric Smelting, 67
BROUGH (B. H.), Mine-Surveying, . 57
BRUCE (Robt.), Food Supply, . . 54
BUCK (R. C.), Algebra, . . . 42
—— Trigonometry, ... 42
BURNS (D.), Colliery Electricity, . 58
BUTTERFIELD (W. J. A.), Gas, . 77
CAIN & THORPE, Synthetic Dyestuffs, 82
CARTER (H. R.), Long Fibre Spinning, 83
CASTELL-EVANS, Chemical Tables, 79
CLAYWORKER'S HANDBOOK, . 75
COLE (Prof G. A. J.), Practical Geology, 53
—— Open Air Studies in Geology, . . 85
COLE (W. H.), Light Railways, . 30
COLLINS, (H. F.), Lead and Silver, . 64
COX (S. H.), Prospecting for Minerals, 55
CRIMP (W. S.), Sewage Disposal Works, 76
CUNNINGHAM (B.), Dock Engineering, 27
DAVEY (H.), Pumping Machinery, . 37
DIXON (C.), Bird-Life, . . . 85
DONKIN (Bryan), Gas and Oil Engines, 28
—— Efficiency of Steam Boilers, . 28
DOWSING (H. J.), Electrical Prices, 49
DUERR (Geo.), Bleaching, &c., . . 84
DUPRÉ & HAKE, Manual of Chemistry, 70
EICHHORN, Wireless Telegraphy, . 49
FARNSWORTH, Cons. Steel Work, . 26
FIDLER (Prof.), Bridge-Construction, 26
FIELD (E. R.), Mining Report Book, . 57
FOSTER (Sir C. Le Neve), Mining, . 56
—— Elementary Mining, . . . 56
—— & HALDANE, Mine Air, . . 57
GAIRNS (J. F.), Loco. Compounding, 30
GINSBURG (Dr.), Legal Duties of Shipmasters, ... 43
GOLDING (H. A.), Bonus Tables, . 31
GROSSMANN (J.), Chem. Engineering, 46
GURDEN (R.), Traverse Tables, . 31
GUTTMANN (O.), Blasting, . . 58
HANNAN (W. I.), Textile Fibres. . 83
HARBORD (F. W.), Steel, . . . 65
HARRISON (J. W.), Sanitation, . . 79
HICKS (J. A.), Mineral Oil Testing, . 61
HODGSON (R. B.), Emery Grinding, 33
HUGHES (H. W.), Coal Mining, . 56
HURST (Chas.), Valves, . . . 31
—— Hints on Engine Design, . . 31
HURST (G. H.), Painters' Colours, . 80
—— Laboratory Guide to Paints, . 80
—— Garment Dyeing and Cleaning . 84
JAMIESON (Prof.), MANUALS, . . 34
JOHNSON (J. C. F.), Getting Gold, . 58
JUDE (A.), Steam Turbine, . . . 28
JULIAN (H. F.) and SMART (E.), Cyaniding Gold and Silver Ores, . 59
KASSNER (T.), Gold Seeking, . . 59
KERR (G. L.), Practical Coal-Mining, 58
— Elementary Coal-Mining, . . 58
KNECHT & RAWSON, Dyeing, . . 82
LAFAR, Technical Mycology, . . 74

PAGE

LAMBERT, Glue, Gelatine, &c., . . 81
LAWN (J. G.), Mine Accounts, . . 57
LEEDS (F. H.), Acetylene, . . . 77
LIVERSIDGE, Engine-Room Practice, 29
MACKENZIE (T.), Mechanics, . . 42
MACLEOD (W. A.) and WALKER (C.), Metallurgical Chemistry, . 60
M'MILLAN (W. G.), Electro-Metallurgy, 67
—— & BORCHERS, Electric Smelting, 67
M'WILLIAM, Foundry Practice, . 68
MIDDLETON (R. E.), Water Supply, 77
MILL (Dr. R. H.), New Lands, . 54
MILLAR (W. J.), Latitude & Longitude, 43
MITCHELL (C. A.), Flesh Foods, . 74
—— & HEPWORTH (T. C.), Inks, . 81
MORGAN (J. J.), Metallurgical Tables, 66
MUNRO & JAMIESON'S Electrical Pocket-Book, ... 48
MUNRO (R. D.), Steam-Boilers, . . 32
—— Kitchen Boiler Explosions, . . 32
NAYLOR (W.), Trades' Waste, . . 76
NICHOLSON (W.), Smoke Abatement, 76
NORTH (S.), Oil Fuel, ... 29
OPPENHEIMER (C.), Ferments, . 74
—— Toxines and Antitoxines, . . 74
OSMOND & STEAD, Micro. Analysis, 60
PARK, (J.), Cyanide Process, . . 59
—— Mining Geology, ... 55
PEARCE (W. J.), Painting, . . . 80
PETTIGREW (W. F.), Loco. Eng., . 30
PHILLIPS & BAUERMAN, Metallurgy, 60
POYNTING (Prof.), Mean Density, . 50
—— & THOMSON, Physics, . . . 50
PRAEGER (R. L.), Open Air Botany, . 85
RANKINE'S Works, . . . 35, 36
RAWSON, GARDNER, & LAYCOCK, Dictionary of Dyestuffs, . . . 82
REDGRAVE (G. R.), Cements, . . 76
REDWOOD (Dr Boverton), Petroleum, 61
—— & THOMSON, Handbook, . . 61
REED (Sir E. J.), Stability of Ships, . 38
REID (Geo., M.D.), Sanitation, . . 78
REIMER, Shaft Sinking, . . . 58
RICHMOND (H. D.), Dairy Chemistry, 73
—— Dairy Analysis, ... 73
RILEY (W.), Bacteriology of Brewing, 75
ROBERTS-AUSTEN, Metallurgy, . 3
ROBINSON (Prof.), Hydraulics, . . 37
ROSE (T. K.), Gold, Metallurgy of, . 63
ROTHWELL, (C. F. S.), Textile Printing, 83
SCHWACKHÖFER and BROWNE, Fuel and Water, ... 47
SCHWARTZ (Dr. von), Fire Risks, . 77
SEATON (A. E.), Marine Engineering, 44
—— & ROUNTHWAITE, Marine Engineers' Pocket-Book, . . . 44
SEELEY (Prof.), Physical Geology, . 52
SEXTON (Prof.), Metallurgy, . . 66
—— Quantitative & Qualitative Analysis, 70
SHELTON (W. V.), Mechanic's Guide, 36
SINDALL, Paper Technology, . . 81
SMITH, (J.), Shipmaster's Medical Help, 43
SMITH (Prof. R. H.), Calculus, . . 45
—— Measurement Conversions, . . 45
SOLOMON (H. G.), Electricity Meters, 49
STRICKLAND (F.), Petrol Motors, . 33
SUPLEE (H. H.), Mech. Pocket Book, 45
SYKES (Dr. W. J.), Brewing, . . 75
TRAILL (T. W.), Boilers, . . . 29
TURNER (Thos.), Iron, Metallurgy of, 65
—— Ironfounding, ... 68
WALTON (T.), Know Your Own Ship, 44
—— Present-Day Shipbuilding, . . 38
—— Steel Ships, ... 38
WEBER (Dr. C. O.), India Rubber, . 81
WELLS (S. H.), Engineering Drawing, 27
WIGLEY (T. B.), Jeweller's Art, . 68
WILLOUGHBY (Dr. E. F.), Milk, . 73
WOOD (Francis), Sanitary Engineering, 78
WORDINGHAM, Electrical Stations, . 48
WRIGHT (Dr. A.), Oils and Fats, . 71

Griffin's Standard Publications

		PAGE
Applied Mechanics,	Rankine, Browne, Jamieson,	35, 46, 34
Civil Engineering,	Prof. Rankine,	35
Design of Structures,	S. Anglin,	26
Bridge-Construction,	Prof. Fidler,	26
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Dock Engineering,	B. Cunningham,	27
Engineering Drawing,	S. H. Wells,	27
Constructional Steelwork,	A. W. Farnsworth,	26
Central Electrical Stations,	C. H. Wordingham,	48
Electricity Control,	L. Andrews,	48
,, Meters,	H. G. Solomon,	49
Light Railways,	W. H. Cole,	30
Sewage Disposal Works,	Santo Crimp,	76
Sanitary Engineering,	F. Wood,	78
Traverse Tables,	R. L. Gurden,	31
Locomotive Engineering,	W. F. Pettigrew,	30
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Valves and Valve-Gearing,	Chas. Hurst,	31
Hints on Design,	Chas. Hurst,	31
Marine Engineering,	A. E. Seaton,	44
The Steam Turbine,	Alexander Jude,	28
Marine Steam Turbine,	Prof. Biles,	29
Engine-Room Practice,	J. G. Liversidge,	29
Pocket-Book,	Seaton and Rounthwaite,	44
Present Day Shipbuilding,	T. Walton,	38
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Stability of Ships,	Sir E. J. Reed,	38
Nautical Series,	Ed. by Capt. Blackmore,	39
The Steam-Engine,	Rankine, Jamieson,	35, 34
Gas, Oil, and Air-Engines,	Bryan Donkin,	28
Boilers: Land and Marine,	T. W. Traill,	29
,, Steam,	R. D. Munro,	32
,, Kitchen,	R. D. Munro,	32
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Oil Fuel,	Sidney H. North,	29
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Hydraulic Machinery,	Prof. Robinson,	37
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Bonus Tables,	H. A. Golding,	31
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